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tt^*-bundles in para-complex geometry, special para-Kähler manifolds and para-pluriharmonic maps

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Abstract

We introduce the notion of a para-tt^*-bundle, the generalization of a tt^*-bundle (compare [V. Cortés, L. Schäfer, Topological–antitopological fusion equations, pluriharmonic maps and special Kähler manifolds, in: O. Kowalski, E. Musso, D. Perrone (Eds.), Proceedings of the Conference "Curvature in Geometry" organized in Lecce in honor of Lieven Vanhecke, Progress in Mathematics, vol. 234, Birkhäuser, 2005] and [L. Schäfer, tt^*-geometry and pluriharmonic maps, Ann. Global Anal. Geom., in press]) in para-complex geometry. The main result is the definition of a map Φ from the space of metric para-tt^*-bundles of rank r over a para-complex manifold M to the space of para-pluriharmonic maps from M to $GL(r)/O(p,q)$ where (p,q) is the signature of the metric and the description of the image of this map Φ. Then we recall and prove some results known in special complex and special Kähler geometry in the setting of para-complex geometry, which we use in the sequel to give a simple characterization of the tangent bundle of a special para-complex and special para-Kähler manifold as a particular type of tt^*-bundles. For the case of a special para-Kähler manifold it is shown that the para-pluriharmonic map coincides with the dual Gauß map, which is a para-holomorphic map into the symmetric space $Sp(\mathbb{R}^{2n})/U^{\pi}(C^n) \subset SL(2n)/SO(n,n) \subset GL(2n)/O(n,n)$.

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1. Introduction

The complex version of the geometries introduced in this paper, the tt^*-geometries, originated in the physics of topological-field-theories (for further information see, for example, [2]). As we have shown in [4] using differential geometric arguments or as follows from the results of Hertling [7] using variations of Hodge-structures, special complex and special Kähler manifolds are a special class of solutions, which play a crucial role as admissible target spaces in certain super-symmetric field theories (for more about these see [1] and [6]). In [9] we have proven the existence of a map Φ from the space of metric tt^*-bundles of rank r over the complex manifold (M, J) to the space of (twisted) pluriharmonic maps from the complex manifold (M, J) to the pseudo-Riemannian symmetric space $\mathrm{GL}(r, \mathbb{R})/\mathrm{O}(p, q)$ where (p, q) is the signature of the metric and characterized the image of this map. In the positive definite case the map Φ is essentially bijective. For metric tt^*-bundles with positive definite metric on a real form of the holomorphic tangent bundle $T^{1,0}M$ of the manifold (M, J) this result is due to Dubrovin [5]. In the case of a tt^*-bundle coming from a special Kähler manifold this map is essentially the dual Gauß map, as we have shown in [4]. Further we characterized the tt^*-bundles, which come from special Kähler manifolds.

Recently, special para-complex and special para-Kähler geometry was introduced in [3]. It arises as one of the special geometries of Euclidean super-symmetry. The appearance of the para-complex versions of special geometries motivated us to introduce the notion of para-tt^*-bundles. The natural questions, if special para-complex and special para-Kähler geometries are special solutions of tt^*-geometry and if there exists a map to the space of para-pluriharmonic maps from M to $\mathrm{GL}(r, \mathbb{R})/\mathrm{O}(p, q)$, $\mathrm{SL}(r, \mathbb{R})/\mathrm{SO}(p, q)$ respectively, i.e., pluriharmonic maps translated in the para-complex category, arise and are answered positively in this work. In addition, we analyze the para-pluriharmonic map in the context of special geometry and show, that it is closely related to the dual Gauß map.

2. Para-complex geometry

In this section we recall some definitions of special para-complex geometry given in [3] and prove some results which are analogous to these proven for special Kähler manifolds in [1]. We give here just a sketch of the results needed in this paper. The interested reader could find further informations in [3].

2.1. Para-complex manifolds

Definition 1. A *para-complex structure* on a (real) finite dimensional vector space V is a nontrivial involution $\tau \in \mathrm{End}(V)$, i.e., $\tau^2 = \mathrm{Id}_V$ and $\tau \neq \mathrm{Id}_V$, such that the two eigenspaces $V^{\pm} := \ker(\mathrm{Id} \mp \tau)$ of τ have the same dimension. A *para-complex vector space* (V, τ) is a vector space endowed with a para-complex structure τ. A *para-complex subspace* of the para-complex vector space V is a subspace W of the real vector-space V, such that the restriction of τ to W is a para-complex structure.

Definition 2. An *almost para-complex structure* on a smooth manifold M is an endomorphism field $\tau \in \Gamma(\mathrm{End}(TM))$, $p \mapsto \tau_p$, such that τ_p is a para-complex-structure for all $p \in M$. An almost para-complex structure is called *integrable* if the eigendistributions $T^{\pm}M$ are both integrable. An integrable almost para-complex structure is called *para-complex structure*. A manifold with a para-complex structure is called *para-complex manifold*.

We remark, that the integrability of an almost para-complex structure τ is shown in [3] to be equivalent to the vanishing of the Nijenhuis tensor of τ defined by

$$N_\tau(X,Y) := [X,Y] + [\tau X, \tau Y] - \tau[X, \tau Y] - \tau[\tau X, Y], \quad \text{where } X, Y \in \Gamma(TM).$$

Definition 3. A smooth map $f:(M,\tau) \to (N,\tau')$ from a para-complex manifold (M,τ) to a para-complex manifold (N,τ') is called para-holomorphic if $df \circ \tau = \tau' \circ df$ and anti-para-holomorphic if $df \circ \tau = -\tau' \circ df$.

To go further we introduce the algebra C of para-complex numbers. This is the real algebra generated by 1 and the symbol e subject to the relation $e^2 = 1$. If one regards e as a unit vector in a one-dimensional $\mathbb{R}$-vector space with negative definite scalar product, then C is the Clifford algebra $\mathrm{Cl}_{0,1} = \mathbb{R} \oplus \mathbb{R}$ (compare $\mathrm{Cl}_{1,0} = \mathbb{C}$). As for complex numbers we define the para-complex conjugation

$$\bar{\cdot} : C \to C, \qquad x + ey \mapsto x - ey \quad \text{for } x, y \in \mathbb{R}, \tag{2.1}$$

which is a C-anti-linear involution, i.e., $\overline{ez} = -e\bar{z}$.

Real and imaginary parts are defined as

$$x = \operatorname{Re} z := (z + \bar{z})/2 \quad \text{and} \quad y = \operatorname{Im} z := e(z - \bar{z})/2. \tag{2.2}$$

One has $z\bar{z} = x^2 - y^2$.

Every para-complex vector space V is isomorphic to a trivial free C-module C^k for some k. Obviously para-complex sub-spaces $W \subset V$ correspond to free sub-modules of W.

The decomposition of TM over a para-complex manifold M in T^+M and T^-M induces a bigrading on exterior forms

$$\Lambda^k T^*M = \bigoplus_{k=p+q} \Lambda^{p+,q-} T^*M. \tag{2.3}$$

We regard further the *para-complexification* $TM^C = TM \otimes_{\mathbb{R}} C$ of the tangent bundle TM of an almost para-complex manifold (M,τ) and extend $\tau : TM \to TM$ C-linearly to $\tau : TM^C \to TM^C$. Then for all $p \in M$ the free C-module T_pM^C decomposes as C-module into the direct sum of two free C-modules

$$T_pM^C = T_p^{1,0}M \oplus T_p^{0,1}M \tag{2.4}$$

where

$$T_p^{1,0}M := \{X + e\tau X \mid X \in T_pM\} \quad \text{and} \quad T_p^{0,1}M := \{X - e\tau X \mid X \in T_pM\}.$$

The subbundles $T_p^{1,0}M$ and $T_p^{0,1}M$ can be characterized as the $\pm e$-eigenbundles of the map $\tau : TM^C \to TM^C$, i.e., $\tau = e$ on $T^{1,0}M$ and $\tau = -e$ on $T^{0,1}M$.

In the same manner we decompose $T^*M^C = \Lambda^{1,0}T^*M \oplus \Lambda^{0,1}T^*M$ into the $\pm e$-eigenbundles of the dual para-complex structure $\tau^* : T^*M^C \to T^*M^C$. This decomposition induces a bi-grading on the C-valued exterior forms

$$\Lambda^k T^*M^C = \bigoplus_{p+q=k} \Lambda^{p,q}\, T^*M$$

and finally on the C-valued differential forms on M

$$\Omega^k_C(M) = \bigoplus_{p+q=k} \Omega^{p,q}(M).$$

In the complex case it is well known, that every complex manifold admits a complex torsion-free connection (see for example [8]). We now generalize this theorem to the para-complex case:

Theorem 1. *Every almost para-complex manifold (M,τ) admits an almost para-complex affine connection with torsion*

$$N_\tau = -4T,$$

where N_τ is the Nijenhuis-tensor of the almost para-complex structure τ.

Proof. Let ∇ be a torsion-free connection on M. We define $Q \in \Gamma((T^*M)^2 \otimes TM)$ as

$$4Q(X,Y) := \left[(\nabla_{\tau Y}\tau)X + \tau\big((\nabla_Y\tau)X\big) + 2\tau\big((\nabla_X\tau)Y\big)\right]$$

and further

$$\tilde\nabla_X Y = \nabla_X Y + Q(X,Y).$$

Now we compute

$$\begin{aligned}(\tilde\nabla_X\tau)Y &= \tilde\nabla_X\tau Y - \tau\tilde\nabla_X Y = \nabla_X\tau Y + Q(X,\tau Y) - \tau\nabla_X Y - \tau Q(X,Y)\\ &= (\nabla_X\tau)Y + \underbrace{\big(Q(X,\tau Y) - \tau Q(X,Y)\big)}_{=:A(X,Y)}.\end{aligned}$$

Hence we have to show $A(X,Y) = -(\nabla_X\tau)Y$. It is

$$\begin{aligned}4Q(X,\tau Y) &= (\nabla_Y\tau)X + \tau\big((\nabla_{\tau Y}\tau)X\big) + 2\tau\big((\nabla_X\tau)\,\tau Y\big),\\ 4\tau Q(X,Y) &= \tau(\nabla_{\tau Y}\tau)X + \big((\nabla_Y\tau)X\big) + 2\big((\nabla_X\tau)Y\big).\end{aligned}$$

With $\tau^2 = \mathrm{id}$ we get $\tau[(\nabla_X\tau)\tau Y] = -\tau[\tau(\nabla_X\tau)Y] = -(\nabla_X\tau)Y$ and finally

$$4A = 4\big(Q(X,\tau Y) - \tau Q(X,Y)\big) = -4(\nabla_X\tau)Y.$$

It remains to compute the torsion of $\tilde\nabla$:

$$T^{\tilde\nabla}_{X,Y} = T^{\nabla}_{X,Y} + Q(X,Y) - Q(Y,X) = Q(X,Y) - Q(Y,X).$$

With the definition of Q we find

$$
\begin{aligned}
4T^{\tilde{\nabla}}_{X,Y} &= (\nabla_{\tau Y}\tau)X + \tau\big((\nabla_Y\tau)X\big) + 2\tau\big((\nabla_X\tau)Y\big) \\
&\quad - \big((\nabla_{\tau X}\tau)Y + \tau\big((\nabla_X\tau)Y\big) + 2\tau\big((\nabla_Y\tau)X\big)\big) \\
&= (\nabla_{\tau Y}\tau)X - (\nabla_{\tau X}\tau)Y + \tau\big((\nabla_X\tau)Y\big) - \tau\big((\nabla_Y\tau)X\big) \\
&= (\nabla_{\tau Y}\tau X) - (\nabla_{\tau X}\tau Y) - \tau(\nabla_{\tau Y}X - \nabla_{\tau X}Y) \\
&\quad + \tau\big[\nabla_X(\tau Y) - \tau\nabla_X Y\big] - \tau\big[\nabla_Y(\tau X) - \tau\nabla_Y X\big] \\
&= [\tau Y, \tau X] + [Y, X] + \tau[\nabla_X(\tau Y) - \nabla_{\tau Y}X] + \tau[\nabla_{\tau X}Y - \nabla_Y\tau X] \\
&= [\tau Y, \tau X] + [Y, X] - \tau[\tau Y, X] - \tau[Y, \tau X] \\
&= N_\tau(Y, X) = -N_\tau(X, Y). \qquad \square
\end{aligned}
$$

Corollary 1. *Every para-complex manifold (M, τ) admits a para-complex torsion-free affine connection.*

2.2. *Para-Kähler manifolds*

In this subsection we recall some definitions and results from [3].

Definition 4. Let (V, τ) be a para-complex vector space. A *para-Hermitian* scalar product g on V is a pseudo-Euclidean scalar product for which τ is an anti-isometry, i.e.,

$$\tau^* g = g(\tau\cdot, \tau\cdot) = -g(\cdot, \cdot).$$

A *para-Hermitian vector space* is a para-complex vector space endowed with a para-Hermitian scalar product. The pair (τ, g) is called *para-Hermitian structure* on the vector space V.

Definition 5. An *almost para-Hermitian manifold* (M, τ, g) is an almost para-complex manifold (M, τ) endowed with a pseudo-Riemannian metric g such that $\tau^* g = -g$. If τ is integrable, we call (M, τ, g) *para-Hermitian manifold*. The two-form $\omega := g(\tau\cdot, \cdot)$ is called the *fundamental two-form* of the almost para-Hermitian manifold (M, τ, g).

Definition 6. A *para-Kähler manifold* (M, τ, g) is a para-Hermitian manifold such that τ is parallel with respect to the Levi-Civita-connection D of g, i.e., $D\tau = 0$.

Remark 1. The fundamental two-form satisfies $\tau^*\omega = -\omega$ and hence is of type $(1, 1)$ (considered as C-valued two-form).

Since $D\tau = 0$ implies $N_\tau = 0$ and $d\omega = 0$, any para-Kählerian manifold is a para-Hermitian manifold with closed fundamental two-form. On a para-Kähler-manifold ω is called *para-Kähler-form*. In fact, para-Kähler-manifolds are characterized in [3] to be para-Hermitian manifolds with closed fundamental two-form.

2.3. *Affine special para-complex and special para-Kähler manifolds*

Definition 7. An *affine special para-complex manifold* (M, τ, ∇) is a para-complex manifold (M, τ) endowed with a torsion-free flat connection such that $\nabla\tau$ is a symmetric $(1, 2)$-tensor field, i.e., $(\nabla_X\tau)Y = (\nabla_Y\tau)X$ for all $X, Y \in TM$.

An *affine special para-Kähler manifold* (M, τ, g, ∇) is a special para-complex manifold (M, τ, ∇), such that (M, τ, g) is a para-Kähler manifold and ∇ is symplectic, i.e., $\nabla\omega = 0$.

Since projective special and projective special para-Kähler manifolds do not occur in this text, we omit the adjective affine. The definition of a special para-Kähler manifold can be found in [3].

In the following part of this subsection we are going to generalize some results to para-complex geometry, which are known from the affine special and the affine special Kähler case (see [1]).

Remark 2. Given a linear connection ∇ on the tangent bundle of a manifold M and an invertible endomorphism field $A \in \Gamma(\mathrm{End}(TM))$ we define the connection

$$\nabla^{(A)} X = A\nabla(A^{-1}X).$$

Given a linear flat connection on the real tangent bundle of a para-complex manifold (M, τ), we define a one-parameter-family of connections by

$$\nabla^\theta = \nabla^{(e^{\theta\tau})} = \nabla^{(\cosh(\theta)\,\mathrm{Id} + \sinh(\theta)\tau)} \quad \text{for } \theta \in \mathbb{R}. \tag{2.5}$$

This family of connections is flat, since:

$$\nabla X = 0 \quad \Leftrightarrow \quad \nabla^\theta(e^{\theta\tau}X) = 0,$$

where X is a local vector field on M.

Lemma 1. *Let ∇ be a flat connection with torsion T on a para-complex manifold (M, τ). Then it is*

$$\nabla^\theta = \nabla + A^\theta, \quad \text{where } A^\theta = e^{\theta\tau}\nabla(e^{-\theta\tau}) = -\sinh(\theta)e^{\theta\tau}\nabla\tau$$

and the torsion T^θ of the connection ∇^θ is given by

$$T^\theta = T + \mathrm{alt}(A^\theta) = T - \sinh(\theta)e^{\theta\tau}d^\nabla\tau. \tag{2.6}$$

Proposition 1. *Let ∇ be a flat torsion-free connection on a para-complex manifold (M, τ). Then the triple (M, τ, ∇) defines a special para-complex manifold if and only if one of the following conditions holds*:

(a) $d^\nabla\tau = 0$.
(b) *The flat connection ∇^θ is torsion-free for some $\theta \neq 0$.*
(b)′ *The flat connection ∇^θ is torsion-free for all $\theta \neq 0$.*
(c) *There exists $0 \neq \theta \in \mathbb{R}$ such that $[e^{\theta\tau}X, e^{\theta\tau}Y] = 0$ for all ∇-parallel local vector fields X and Y on M.*
(c)′ $[e^{\theta\tau}X, e^{\theta\tau}Y] = 0$ *for all $\theta \neq 0$ and for all ∇-parallel local vector fields X and Y on M.*
(d) *There exists $0 \neq \theta \in \mathbb{R}$ such that $d(\eta \circ e^{-\theta\tau}) = 0$ for all ∇-parallel local 1-forms on M.*
(d)′ $d(\eta \circ e^{-\theta\tau}) = 0$ *for all $\theta \neq 0$ and for all ∇-parallel local 1-forms on M.*

Proof. The property (a) defines special para-complex manifolds.

As ∇ is torsion-free, the torsion of ∇^θ is by Eq. (2.6):

$$T^\theta = -\sinh(\theta)e^{\theta\tau}d^\nabla\tau.$$

Since $\sinh(\theta) \neq 0$ for $\theta \neq 0$, we get the equivalence of (a) and (b) respectively (b)′.

Let X and Y be ∇-parallel local vector fields. Then $e^{\theta\tau}X$ and $e^{\theta\tau}Y$ are ∇^θ-parallel, by the definition of ∇^θ. Therefore

$$T^\theta(e^{\theta\tau}X, e^{\theta\tau}Y) = [e^{\theta\tau}X, e^{\theta\tau}Y].$$

This gives (b) $\Leftrightarrow$ (c) and (b)$'$ $\Leftrightarrow$ (c)$'$.

For a ∇-parallel 1-form η and X, Y as before we compute:

$$\begin{aligned} d(\eta\circ e^{-\theta\tau})(e^{\theta\tau}X, e^{\theta\tau}Y) &= e^{\theta\tau}X\eta(Y) - e^{\theta\tau}Y\eta(X) - \eta\big(e^{-\theta\tau}[e^{\theta\tau}X, e^{\theta\tau}Y]\big) \\ &= -\eta\big(e^{-\theta\tau}[e^{\theta\tau}X, e^{\theta\tau}Y]\big), \end{aligned}$$

as the functions $\eta(X)$ and $\eta(Y)$ are constant. This proves (c) $\Leftrightarrow$ (d) and (c)$'$ $\Leftrightarrow$ (d)$'$. $\square$

Proposition 2. *If (M, τ, ∇) is a special para-complex manifold, then (M, τ, ∇^θ) is a special para-complex manifold for any θ.*

If (M, τ, g, ∇) is a special para-Kähler manifold, then $(M, \tau, g, \nabla^\theta)$ is a special para-Kähler manifold for any θ.

Proof. From above we know, that ∇^θ is flat and torsion-free.

In order to prove this proposition we compute $\nabla^\theta\tau$ and $\nabla^\theta\omega$.

Let $X, Y, Z \in \Gamma(TM)$

$$\begin{aligned} (\nabla^\theta_X\tau)Y &= \nabla^\theta_X(\tau Y) - \tau\nabla^\theta_X Y = e^{\theta\tau}\nabla_X(e^{-\theta\tau}\tau Y) - \tau e^{\theta\tau}\nabla_X(e^{-\theta\tau}Y) \\ &= e^{\theta\tau}\nabla_X(\tau e^{-\theta\tau}Y) - e^{\theta\tau}\tau\nabla_X(e^{-\theta\tau}Y) = e^{\theta\tau}(\nabla_X\tau)e^{-\theta\tau}Y \overset{(*)}{=} e^{2\theta\tau}(\nabla_X\tau)Y. \end{aligned}$$

In $(*)$ we have used $\tau(\nabla\tau) = -(\nabla\tau)\tau$, which follows from $\tau^2 = \mathrm{Id}$.

This shows $d^{\nabla^\theta}\tau = e^{2\theta\tau}d^\nabla\tau = 0$.

Further we find utilizing $\omega(\cdot, e^{\theta\tau}\cdot) = \omega(e^{-\theta\tau}\cdot, \cdot)$, which is a consequence of $\tau^*\omega = -\omega$

$$\begin{aligned} \nabla^\theta_Z\omega(X, Y) &= Z.\omega(X, Y) - \omega(\nabla^\theta_Z X, Y) - \omega(X, \nabla^\theta_Z Y) \\ &= Z.\omega(X, Y) - \omega(e^{\theta\tau}\nabla_Z e^{-\theta\tau}X, Y) - \omega(X, e^{\theta\tau}\nabla_Z e^{-\theta\tau}Y) \\ &= Z.\omega(X, Y) - \omega(\nabla_Z e^{-\theta\tau}X, e^{-\theta\tau}Y) - \omega(e^{-\theta\tau}X, \nabla_Z e^{-\theta\tau}Y) \\ &= Z.\omega(X, Y) - Z.\omega(e^{-\theta\tau}X, e^{-\theta\tau}Y) = 0. \qquad \square \end{aligned}$$

Given a para-complex manifold with a flat connection ∇, we define the conjugate connection via

$$\nabla^c_X Y = \nabla^{(\tau)}_X Y = \nabla_X Y + \tau(\nabla_X\tau)Y = \tau\nabla_X(\tau Y) \quad \text{for } X, Y \in \Gamma(TM).$$

Proposition 3. *Let (M, τ) be a para-complex manifold with a torsion-free flat connection ∇. Then the following statements are equivalent*:

(a) *(M, τ, ∇) is a special para-complex manifold.*
(b) *The conjugate connection ∇^c is torsion-free.*

Proof. The torsion of ∇^c is

$$T^{\nabla^c} = T^{\nabla} + \mathrm{alt}\big(\tau(\nabla\tau)\big) = \tau\, d^{\nabla}\tau.$$

Therefore ∇^c is torsion-free if and only if $d^{\nabla}\tau = 0$. $\square$

Proposition 4. *Let (M, τ, ∇) be a special para-complex manifold. Then $D := \frac{1}{2}(\nabla + \nabla^c)$ defines a torsion-free connection such that $D\tau = 0$.*

Proof. As it is a convex-combination of torsion-free connections, D is a torsion-free connection. For any $X \in \Gamma(TM)$ we compute:

$$D_X\tau = \nabla_X\tau + \frac{1}{2}[\tau\nabla_X\tau, \tau] = \nabla_X\tau - \nabla_X\tau = 0. \qquad \square$$

Proposition 5. *Let (M, τ, g, ∇) be a special para-Kähler manifold and ∇^g the Levi-Civita connection of g. Then the following hold*:

(i) $\nabla^g = \frac{1}{2}(\nabla + \nabla^c) = D$.
(ii) *The conjugate connection ∇^c is g-dual, i.e.*:

$$X.g(Y, Z) = g(\nabla^c_X Y, Z) + g(Y, \nabla_X Z)$$

equivalently

$$X.g(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla^c_X Z)$$

for all vector fields $X, Y, Z \in \Gamma(TM)$.
(iii) *The tensor ∇g is completely symmetric.*

Proof. (i) follows immediately from (ii) and Proposition 4.

(ii) follows from a direct calculation which only uses the fact that ω is ∇-parallel and τ-anti-invariant: With $X, Y, Z \in \Gamma(TM)$ one finds

$$\begin{aligned} X.g(Y, Z) &= X.\omega(\tau Y, Z) \\ &= \omega(\nabla_X\tau Y, Z) + \omega(\tau Y, \nabla_X Z) \\ &= -\omega(\tau\nabla_X\tau Y, \tau Z) + g(Y, \nabla_X Z) \\ &= \omega(\tau Z, \tau\nabla_X\tau Y) + g(Y, \nabla_X Z) \\ &= g\big(Z, \tau\nabla_X(\tau Y)\big) + g(Y, \nabla_X Z) \\ &= g(\nabla^c_X Y, Z) + g(Y, \nabla_X Z). \end{aligned}$$

Finally (iii) follows from (ii) with the same argument as in [1]. $\square$

Proposition 6. *Let (M, τ, g, ∇) be a special para-Kähler manifold and D the Levi-Civita connection of g. Define the endomorphism field S as*

$$S := \nabla - D = \nabla - \frac{1}{2}(\nabla + \nabla^c) = \frac{1}{2}(\nabla - \nabla^c) = -\frac{1}{2}\tau(\nabla\tau).$$

Then S is

(i) *symmetric, i.e.*: $S_X Y = S_Y X$; $\forall X, Y \in \Gamma(TM)$,
(ii) ω*-skew-symmetric, i.e.*: $\omega(S\cdot,\cdot) = -\omega(\cdot, S\cdot)$,
(iii) g*-symmetric, i.e.*: $g(S\cdot,\cdot) = g(\cdot, S\cdot)$
(iv) *and anticommutes with* τ, *i.e.*:

$$\{S_X, \tau\} = S_X\tau + \tau S_X = 0 \quad \textit{for all } X \in \Gamma(TM). \tag{2.7}$$

Proof. Let $X, Y, Z \in \Gamma(TM)$.

(i) For a special para-complex manifold ∇ and ∇^c are torsion-free (by definition and Proposition 3), so $\nabla - \nabla^c = -\tau(\nabla\tau) = 2S$ is symmetric.

(ii) In fact $Dg = 0$ (Proposition 5) and $D\tau = 0$ (Proposition 4) imply $D\omega = 0$. In addition $\nabla\omega = 0$ yields

$$\omega(S_X Y, Z) + \omega(Y, S_X Z) = \omega\big((\nabla - D)_X Y, Z\big) + \omega\big(Y, (\nabla - D)_X Z\big) = (\nabla - D)_X \omega(Y, Z) = 0.$$

(iii) Using $X.g(Y,Z) - g(\nabla_X Y, Z) = g(Y, \nabla^c_X Z)$ we prove the g-symmetry of S

$$\begin{aligned} 2g(S_X Y, Z) = g\big((\nabla - \nabla^c)_X Y, Z\big) &= g(\nabla_X Y, Z) - g(\nabla^c_X Y, Z) \\ &= X.g(Y,Z) - g(Y, \nabla^c_X Z) - X.g(Y,Z) + g(Y, \nabla_X Z) \\ &= g\big(Y, (\nabla - \nabla^c)_X Z\big) = 2g(Y, S_X Z). \end{aligned}$$

(iv) Now we need only the ω-skew-symmetry of S, the g-symmetry of S and $\omega = g(\tau\cdot,\cdot) = -g(\cdot, \tau\cdot)$ to get with $X, Y, Z \in \Gamma(TM)$

$$g(S_X\tau Y, Z) = g(\tau Y, S_X Z) = \omega(Y, S_X Z) = -\omega(S_X Y, Z) = -g(\tau S_X Y, Z)$$

and consequently $\{S_X, \tau\} = 0$. $\square$

3. Para-tt^*-bundles

In this section we introduce the definition of a para-complex version of tt^*-bundles which will be referred to as para-tt^*-bundles.

Definition 8. A *para-tt*-bundle* (*ptt^*-bundle*) (E, D, S) over a para-complex manifold (M, τ) is a real vector bundle $E \to M$ endowed with a connection D and a section S of $T^*M \otimes \operatorname{End} E$ satisfying the *ptt^*-equation*

$$R^\theta = 0 \quad \text{for all } \theta \in \mathbb{R}, \tag{3.1}$$

where R^θ is the curvature tensor of the connection D^θ defined by

$$D^\theta_X := D_X + \cosh(\theta) S_X + \sinh(\theta) S_{\tau X} \quad \text{for all } X \in TM. \tag{3.2}$$

A *metric ptt^*-bundle* (E, D, S, g) is a ptt^*-bundle (E, D, S) endowed with a possibly indefinite D-parallel fiber metric g, such that S is g-symmetric, i.e., for all $p \in M$

$$g(S_X Y, Z) = g(Y, S_X Z) \quad \text{for all } X, Y, Z \in T_pM. \tag{3.3}$$

A *unimodular metric para-tt*-bundle* (E, D, S, g) is a metric ptt^*-bundle (E, D, S, g), such that $\operatorname{tr} S_X = 0$ for all $X \in TM$. An *oriented unimodular metric para-tt*-bundle* (E, D, S, g, or) is a unimodular metric ptt^*-bundle endowed with an orientation or of the bundle E.

Remark 3. (1) If (E, D, S) is a ptt^*-bundle then (E, D, S^θ) is a ptt^*-bundle for all $\theta \in \mathbb{R}$, where

$$S^\theta := D^\theta - D = \cosh(\theta) S + \sinh(\theta) S_\tau. \tag{3.4}$$

The same remark applies to metric ptt^*-bundles.

(2) Notice that an oriented unimodular metric ptt^*-bundle (E, D, S, g, or) carries a canonical metric volume element $\nu \in \Gamma(\wedge^r E^*)$, $r = \operatorname{rk} E$, determined by g and or, which is D^θ-parallel for all $\theta \in \mathbb{R}$.

Proposition 7. *Let E be a real vector bundle over a para-complex manifold (M, τ) endowed with a connection D and a section $S \in \Gamma(T^*M \otimes \operatorname{End} E)$.*

Then (E, D, S) is a ptt^*-bundle if and only if D and S satisfy the following equations:

$$R^D + S \wedge S = 0, \quad S \wedge S \textit{ is of type } (1,1), \quad d^D S = 0 \quad \textit{and} \quad d^D S_\tau = 0.$$

We remark, that in the case $(1, 1)$ and $(1+, 1-)$ the two gradings defined in Section 2.1 coincide.

Proof. To prove the proposition, we have to compute the curvature of D^θ.

Let $X, Y \in \Gamma(TM)$ arbitrary:

$$\begin{aligned} R^\theta_{X,Y} &= R^D_{X,Y} + \big[D_X, \cosh(\theta) S_Y + \sinh(\theta) S_{\tau Y}\big] + \big[\cosh(\theta) S_X + \sinh(\theta) S_{\tau X}, D_Y\big] \\ &\quad + \big[\cosh(\theta) S_X + \sinh(\theta) S_{\tau X}, \cosh(\theta) S_Y + \sinh(\theta) S_{\tau Y}\big] \\ &\quad - \cosh(\theta) S_{[X,Y]} - \sinh(\theta) S_{\tau[X,Y]} \\ &= R^D_{X,Y} + \sinh^2(\theta)[S_{\tau X}, S_{\tau Y}] + \cosh^2(\theta)[S_X, S_Y] \\ &\quad + \cosh(\theta)\sinh(\theta)\big([S_X, S_{\tau Y}] + [S_{\tau X}, S_Y]\big) \\ &\quad + \cosh(\theta)\big([D_X, S_Y] + [S_X, D_Y] - S_{[X,Y]}\big) \\ &\quad + \sinh(\theta)\big([S_{\tau X}, D_Y] + [D_X, S_{\tau Y}] - S_{\tau[X,Y]}\big). \end{aligned}$$

The theorems of addition

$$\sinh(a)\cosh(a) = \frac{1}{2}\sinh(2a), \quad \cosh(a)^2 = \frac{1}{2}\big(1 + \cosh(2a)\big) \quad \text{and}$$
$$\sinh(a)^2 = \frac{1}{2}\big(\cosh(2a) - 1\big)$$

yield

$$\begin{aligned} R^\theta_{X,Y} &= R^D_{X,Y} + \frac{1}{2}\big([S_X, S_Y] - [S_{\tau X}, S_{\tau Y}]\big) \\ &\quad + \sinh(\theta)\big([S_{\tau X}, D_Y] + [D_X, S_{\tau Y}] - S_{\tau[X,Y]}\big) \\ &\quad + \cosh(\theta)\big([D_X, S_Y] + [S_X, D_Y] - S_{[X,Y]}\big) \\ &\quad + \frac{1}{2}\sinh(2\theta)\big([S_X, S_{\tau Y}] + [S_{\tau X}, S_Y]\big) + \frac{1}{2}\cosh(2\theta)\big([S_{\tau X}, S_{\tau Y}] + [S_X, S_Y]\big). \end{aligned}$$

Separation in $1, \sinh(n\theta), \cosh(n\theta)$ with $n = 1, 2$ implies

$$0 = R^D_{X,Y} + \frac{1}{2}\big([S_X, S_Y] - [S_{\tau X}, S_{\tau Y}]\big),$$

$$0 = [S_{\tau X}, D_Y] + [D_X, S_{\tau Y}] - S_{\tau[X,Y]},$$
$$0 = [D_X, S_Y] + [S_X, D_Y] - S_{[X,Y]} \quad \text{and}$$
$$0 = [S_X, S_Y] + [S_{\tau X}, S_{\tau Y}] = [S_{\tau X}, S_Y] + [S_X, S_{\tau Y}]$$

and equivalently

$$S \wedge S(X,Y) = [S_X, S_Y] = -[S_{\tau X}, S_{\tau Y}], \quad \text{i.e., is of type } (1,1),$$
$$R^D + S \wedge S = 0,$$
$$d^D S(X,Y) = D_X(S_Y) - D_Y(S_X) - S_{[X,Y]} = 0 \quad \text{and}$$
$$d^D S_\tau(X,Y) = D_X(S_{\tau Y}) - D_Y(S_{\tau X}) - S_{\tau[X,Y]} = 0. \qquad \square$$

4. Special para-complex and special para-Kähler manifolds as solutions of ptt^*-geometry

In this section we are interested in ptt^*-bundles on the tangent-bundle TM of a para-complex manifold (M, τ). In this context it is natural to regard ptt^*-bundles, for that the connection ∇^θ is torsion-free.

Definition 9. A ptt^*-bundle (TM, D, S) over a para-complex manifold (M, τ) is called *special* if D^θ is torsion-free and special, i.e., $D^\theta \tau$ is symmetric for all θ.

Proposition 8. *A ptt^*-bundle (TM, D, S) is special if and only if D is torsion-free and $D\tau$, S and S_τ are symmetric.*

Proof. The torsion T^θ of D^θ equals

$$T^\theta(X,Y) = T(X,Y) + \cosh(\theta)(S_X Y - S_Y X) + \sinh(\theta)(S_{\tau X} Y - S_{\tau Y} X), \tag{4.1}$$

where T is the torsion-tensor of D. This implies, that $T^\theta = 0$ for all θ if and only if $T = 0$ and S and S_τ are symmetric. The equation

$$\begin{aligned} (D^\theta_X \tau)Y &= (D_X\tau)Y + \cosh(\theta)[S_X, \tau]Y + \sinh(\theta)[S_{\tau X}, \tau]Y \\ &\overset{\{S_X,\tau\}=0}{=} (D_X\tau)Y - 2\cosh(\theta)\tau S_X Y - 2\sinh(\theta)\tau S_{\tau X} Y \end{aligned} \tag{4.2}$$

shows that $D^\theta\tau$ is symmetric if $D\tau$, S and S_τ are symmetric. Conversely, if $T^\theta = 0$ and $D^\theta\tau$ is symmetric, then the first part of the proof yields, that S and S_τ are symmetric and $T = 0$. Eq. (4.2) implies finally the symmetry of $D\tau$. $\square$

Theorem 2.

(i) *Let (M, τ, ∇) be a special para-complex manifold. Put $S := -\frac{1}{2}\tau\nabla\tau$ and $D := \nabla - S$. Then (TM, D, S) is a special ptt^*-bundle with the following additional properties*:
   (a) $S_X\tau = -\tau S_X$ *for all* $X \in TM$ *and*
   (b) $D\tau = 0$.
   This defines a map Φ from special para-complex manifolds to special ptt^*-bundles.

(ii) *Let (TM, D, S) be a special ptt^*-bundle over a para-complex manifold (M, τ). Then $(M, \tau, \nabla := D + S)$ is a special para-complex manifold. This defines a map Ψ from special ptt^*-bundles to special para-complex manifolds such that $\Psi \circ \Phi = \mathrm{Id}$. If (TM, D, S) is a special ptt^*-bundle satisfying the conditions* (a) *and* (b) *of* (i)*, then $\Phi(\Psi(TM, D, S)) = (TM, D, S)$.*

(iii) *Let (M, τ, g, ∇) be a special para-Kähler manifold with S and D as in* (i)*. Then (TM, D, S, g) defines a special metric ptt^*-bundle satisfying* (a) *and* (b) *of* (i)*. This defines a map, also called Φ, from special para-Kähler manifolds to special metric ptt^*-bundles.*

(iv) *Let (TM, D, S, g) be a special metric ptt^*-bundle over a para-Hermitian manifold (M, τ, g) satisfying the conditions* (a) *and* (b) *in* (i)*. Then $(M, \tau, g, \nabla := D + S)$ is a special para-Kähler manifold. In particular, we have a map Ψ from special metric ptt^*-bundles over para-Hermitian manifolds (M, τ, g) satisfying the conditions* (a) *and* (b) *in* (i) *to special para-Kähler manifolds. Moreover Ψ is a bijection and $\Psi^{-1} = \Phi$.*

(v) *Let (TM, D, S, g) be a metric ptt^*-bundle over a para-Hermitian manifold (M, τ, g) satisfying the conditions* (a) *and* (b) *in* (i) *and such that D is torsion-free. Then it is special if and only if $(M, \tau, g, \nabla := D + S)$ is a special para-Kähler manifold.*

Proof. (i) Let (M, τ, ∇) be a special para-complex manifold with S and D defined as above. Then $\nabla^\theta = e^{\theta\tau} \circ \nabla \circ e^{-\theta\tau}$ defines a family of torsion-free flat connections. Using $\nabla = D + S$ we obtain

$$\nabla^\theta_X = D_X + e^{2\theta\tau} S_X,$$

due to the following computation (using $D\tau = 0$ due to Proposition 4 and $\{S_X, \tau\} = 0$ for all $X \in TM$, see Proposition 6)

$$\begin{aligned}
\nabla^\theta_X Y &= e^{\theta\tau}(D_X + S_X)(e^{-\theta\tau} Y) = e^{\theta\tau} D_X\big(\cosh(\theta)Y - \sinh(\theta)\tau Y\big) + e^{\theta\tau} S_X(e^{-\theta\tau} Y) \\
&\overset{\{\tau, S_X\}=0}{=} e^{\theta\tau}\big(\cosh(\theta) D_X Y - \sinh(\theta)\big)\big((D_X\tau)Y + \tau D_X Y\big) + e^{2\theta\tau} S_X Y \\
&= D_X Y + e^{2\theta\tau} S_X Y.
\end{aligned}$$

Now we show that $\nabla^\theta = D^{-2\theta}$, where D^θ was defined in (3.2):

$$\begin{aligned}
\nabla^\theta_X - D_X &= e^{2\theta\tau} S_X = \cosh(2\theta) S_X + \sinh(2\theta)\tau S_X \\
&\overset{(*)}{=} \cosh(2\theta) S_X - \sinh(2\theta) S_{\tau X} = D^{-2\theta}_X - D_X, \quad X \in TM.
\end{aligned}$$

In $(*)$ we have used that $\tau S_X = -S_{\tau X}$, which follows from the symmetry of S and $\{S, \tau\} = 0$ by

$$\tau S_X Y = \tau S_Y X = -S_Y \tau X = -S_{\tau X} Y, \quad X, Y \in T_p M.$$

This shows, that (TM, D, S) is a special ptt^*-bundle. The additional properties hold, as (M, τ, ∇) is a special para-complex manifold (compare Propositions 4 and 6).

(ii) In order to prove the second statement, let (TM, D, S) be a special ptt^*-bundle, i.e., D^θ is flat, torsion-free and special. In particular, $\nabla = D + S = D^0$ is flat, torsion-free and special. Hence (M, τ, ∇) is a special para-complex manifold. Obviously we have $\Psi \circ \Phi = \mathrm{Id}$.

Conversely, let (TM, D, S) be a special ptt^*-bundle satisfying $D\tau = 0$ and $S_X\tau = -\tau S_X$ for all $X \in T_p M$. Then we obtain D and S from $\nabla = D + S$ by the formulas $S = -\frac{1}{2}\tau\nabla\tau$ and $D = \nabla - S$. In fact, let (TM, D', S') be another special ptt^*-bundle over (M, τ) with $D'\tau = 0$, $S'_X\tau = -\tau S'_X$ for all $X \in T_p M$ and $\nabla = D + S = D' + S'$ then

$$0 = D'_X\tau = \nabla_X\tau - [S'_X, \tau] = \nabla_X\tau + 2\tau S'_X \tag{4.3}$$

for all $X \in T_pM$. This shows $S'_X = -\frac{1}{2}\tau\nabla\tau = S_X$ and $D' = \nabla - S' = \nabla - S = D$.

(iii) Let (M, τ, g, ∇) be a special para-Kähler manifold with D and S defined as in (i). Then (TM, D, S) is a special ptt^*-bundle satisfying (a) and (b), due to (i). Proposition 5 implies, that $Dg = 0$ and Proposition 6 implies, that S is g-symmetric and hence that (TM, D, S, g) is a special metric ptt^*-bundle.

(iv) Let (TM, D, S, g) be a special metric ptt^*-bundle over a para-Hermitian manifold (M, τ, g) satisfying (a) and (b) in (i). By (ii), we know already, that $(M, \tau, \nabla := D + S)$ is a special para-complex manifold. Therefore it remains to prove $\nabla\omega = 0$. We have $Dg = 0$ and $D\tau = 0$ (property (b) in (i)) and consequently $D\omega = 0$. As $D\omega = 0$, $\nabla\omega = 0$ is equivalent to the ω-skew-symmetry of S and finally to the g-symmetry of S, since $\{\tau, S_X\} = 0$. But by the definition of a metric ptt^*-bundle S is g-symmetric. Therefore (M, τ, ∇, g) is a special para-Kähler manifold. The rest of part (iv) follows from part (ii).

(v) We have only to show the direction which follows not from (iv). Let (TM, D, S, g) be a metric ptt^*-bundle over a para-Hermitian manifold (M, τ, g), such that $(M, \tau, g, \nabla = D + S) = \Psi(TM, D, S, g)$ is a special para-Kähler manifold. If D is torsion-free, then it is the Levi-Civita connection of g, and therefore $D = \nabla + \frac{1}{2}\tau\nabla\tau$, see Proposition 5. This shows, that $\Phi(M, \tau, g, \nabla) = (TM, D, S, g)$ and that (TM, D, S, g) is a special metric ptt^*-bundle. □

Corollary 2. *A metric ptt^*-bundle (TM, D, S, g) over a para-Hermitian manifold (M, τ, g) which satisfies* (a) *and* (b) *in Theorem 2 is oriented and unimodular.*

Proof. By Theorem 2, $(M, \tau, g, \nabla = D + S)$ is a special para-Kähler manifold. Hence we can orient it by $\omega \wedge \cdots \wedge \omega$, where ω is its para-Kähler-form. Its para-Kähler-form is parallel with respect to the connections D and ∇ and therefore invariant under $S_X = \nabla_X - D_X$. This shows $\operatorname{tr} S_X = 0$. □

5. Para-pluriharmonic maps

In this section we introduce the notion of para-pluriharmonic maps as an analogue to pluriharmonic maps and deduce some properties of para-pluriharmonic maps, especially to $S(p, q) = \mathrm{GL}(r)/\mathrm{O}(p, q)$[2] and $S^1(p, q) = \mathrm{SL}(r)/\mathrm{SO}(p, q)$ which are needed later and which we have not found in the literature.

Definition 10. Let (M, τ) be a para-complex manifold and (N, h) be a pseudo-Riemannian manifold with Levi-Civita connection ∇^h, D a connection on M which satisfies

$$D_{\tau Y}X = \tau D_Y X \tag{5.1}$$

for all vector fields which satisfy $\mathcal{L}_X\tau = 0$ (i.e., for which $X + e\tau X$ is para-holomorphic) and ∇ the connection on $T^*M \otimes f^*TN$ which is induced by D and ∇^h.

A map $f : M \to N$ is para-pluriharmonic if and only if it satisfies the equation

$$\nabla''\partial f = 0, \tag{5.2}$$

[2] Where $\mathrm{O}(p, q)$ and $\mathrm{SO}(p, q)$ are the pseudo-orthogonal groups of signature (p, q).

where $\partial f = df^{1,0} \in \Gamma(\bigwedge^{1,0} T^*M \otimes_C (TN)^C)$ is the $(1,0)$-component of $d^C f$ and ∇'' is the $(0,1)$-component of $\nabla = \nabla' + \nabla''$.

Equivalently one regards $\alpha = \nabla d\phi \in \Gamma(T^*M \otimes T^*M \otimes \phi^* TN)$.

Then ϕ is para-pluriharmonic if and only if

$$\alpha(X,Y) - \alpha(\tau X, \tau Y) = 0$$

for all $X, Y \in TM$. This can also be expressed as

$$\alpha^{1+,1-} = 0.$$

We recall, that in the case $(1,1)$ and $(1+,1-)$ the two gradings defined in Section 2.1 coincide.

On every para-complex manifold exists a para-complex torsion-free connection, as we have shown in Corollary 1. The following proposition ensures now the existence of a connection satisfying Eq. (5.1).

Proposition 9.

(i) *Every para-complex torsion-free connection D on a para-complex manifold (M,τ) satisfies Eq. (5.1).*
(ii) *On every para-complex manifold there exists a connection satisfying Eq. (5.1).*

Proof. (i) The conditions $T^D = 0$ and $D\tau = 0$ yield

$$D_{\tau Y} X - \tau D_Y X = [\tau Y, X] + D_X(\tau Y) - \tau D_Y X = [\tau Y, X] - \tau[Y, X] = -(\mathcal{L}_X \tau)Y. \tag{5.3}$$

The right-hand side vanishes if $\mathcal{L}_X \tau = 0$.

(ii) The existence of a para-complex torsion-free connection D on (M,τ) follows from Corollary 1. Part (i) implies now the statement (ii). □

This result motivates the

Convention: In the following text the connection D on M is taken to be torsion-free and para-complex.

Definition 11. A *para-complex curve* is a para-complex manifold of para-complex dimension one. A *para-complex curve in a para-complex manifold M* is a para-complex curve which is a para-complex submanifold of M.

With this notation we have

Proposition 10. *A map $f:(M,\tau) \to (N,h)$ from a para-complex manifold (M,τ) to a pseudo-Riemannian manifold (N,h) is para-pluriharmonic if and only if the restriction of f to any para-complex curve C in M is harmonic.*

Proof. Let $C \subset M$ be a para-complex curve in (M,τ). On C a Hermitian metric g in the para-conformal class of τ is chosen. As g is Hermitian it is of type $(1+,1-)$. Hence the trace of $\nabla df_{|C}$ with respect to g is zero if and only if $\nabla''\partial f_{|C} = 0$, as ∇df is symmetric. Since this holds for all curves C in M the proposition is proven. □

Let $\mathrm{Sym}_{p,q}(\mathbb{R}^r)$ be the space of symmetric $r \times r$ matrices of signature (p,q). These define pseudo-scalar-products of same signature by $\langle\cdot,\cdot\rangle_A = \langle A\cdot,\cdot\rangle_{\mathbb{R}^r}$, where $\langle\cdot,\cdot\rangle_{\mathbb{R}^r}$ is the Euclidean scalar-product. The action of an element $g \in \mathrm{GL}(r)$ is induced by the action of $\mathrm{GL}(r)$ on the pseudo-scalar-products of signature (p,q), i.e., by $g.\langle\cdot,\cdot\rangle_A = \langle g^{-1}\cdot, g^{-1}\cdot\rangle_A = \langle (g^{-1})^t A g^{-1}\cdot,\cdot\rangle_{\mathbb{R}^r}$. This gives an action of $\mathrm{GL}(r)$ $A \mapsto (g^{-1})^t A g^{-1}$ on $\mathrm{Sym}_{p,q}(\mathbb{R}^r)$ which we use to identify $\mathrm{Sym}_{p,q}(\mathbb{R}^r)$ with $S(p,q)$ in the following proposition.

In the same manner, we can identify $S^1(p,q) = \mathrm{SL}(r)/\mathrm{SO}(p,q)$ with the space of symmetric $r \times r$ matrices of signature (p,q) and determinant $(-1)^q$.

Proposition 11.

(a) *Let Ψ be the above identification $\Psi : S(p,q) \tilde{\to} \mathrm{Sym}_{p,q}(\mathbb{R}^r) \subset \mathrm{GL}(r)$ where $\mathrm{GL}(r)$ carries the pseudo-Riemannian metric, given by the (Ad-invariant) trace-form, i.e., $(A,B) \mapsto \mathrm{tr}(AB)$. Then Ψ is a totally-geodesic immersion and a map $\tilde{f}$ from a para-complex manifold (M,τ) to $S(p,q)$ is para-pluriharmonic, iff the map $f = \Psi \circ \tilde{f} : M \to S(p,q) \tilde{\to} \mathrm{Sym}_{p,q}(\mathbb{R}^r) \subset \mathrm{GL}(r)$ is para-pluriharmonic.*

(b) *Let Ψ^0 be the above identification $\Psi^0 : S^1(p,q) \tilde{\to} \mathrm{Sym}^1_{p,q}(\mathbb{R}^r) \subset \mathrm{SL}(r)$ where $\mathrm{SL}(r)$ carries the pseudo-Riemannian metric, given by the (Ad-invariant) trace-form, i.e., $(A,B) \mapsto \mathrm{tr}(AB)$. Then Ψ^0 is a totally-geodesic immersion and a map $\tilde{f}$ from a para-complex manifold (M,τ) to $S^1(p,q)$ is para-pluriharmonic, iff the map $f = \Psi^0 \circ \tilde{f} : M \to S^1(p,q) \tilde{\to} \mathrm{Sym}^1_{p,q}(\mathbb{R}^r) \subset \mathrm{SL}(r)$ is para-pluriharmonic.*

Proof. The proof of the first part was done in [9] by expressing the map Ψ in terms of the well-known Cartan-immersion.

It remains to prove $\tilde{f}$ para-pluriharmonic if and only if $\Psi \circ \tilde{f}$ is para-pluriharmonic.

Let X, Y be para-holomorphic vector-fields and D the connection on M. We calculate $\nabla df(X,Y) = \nabla_X(df(Y)) - df(D_X Y)$. Taking the $(1,1)$-part of this expression we find for the second term:

$$df(D_X Y - D_{\tau X}\tau Y) = df(D_X Y - \tau^2 D_X Y) = 0.$$

Therefore the para-pluriharmonicity is independent of the connection chosen on M. Hence we have only to regard the Levi-Civita-connections on $G = \mathrm{GL}(r), \mathrm{SL}(r)$ respectively $G/K = \mathrm{GL}(r)/\mathrm{O}(p,q)$, $\mathrm{SL}(r)/\mathrm{SO}(p,q)$. Let $X, Y \in \Gamma(TM)$ be para-holomorphic and calculate:

$$\nabla^G_X d(\Psi \circ f)(Y) = \nabla^G_X d\Psi\big(df(Y)\big) = \nabla^G_X \Psi_*\big(df(Y)\big) = \Psi_*\big(\nabla^{G/K}_X df(Y)\big) + II(X,Y)$$

where II is the second fundamental-form which vanishes, as the immersion is totally geodesic. This implies

$$\begin{aligned}\alpha^G(X,Y) - \alpha^G(\tau X, \tau Y) &= \nabla^G_X d(\Psi \circ f)(Y) - \nabla^G_{\tau X} d(\Psi \circ f)(\tau Y)\\ &= \Psi_*\big(\nabla^{G/K}_X df(Y) - \nabla^{G/K}_{\tau X} df(\tau Y)\big)\\ &= \Psi_*\big(\alpha^{G/K}(X,Y) - \alpha^{G/K}(\tau X, \tau Y)\big).\end{aligned}$$

Since Ψ is an immersion, the proof is finished. □

Remark 4 (*see also* [4,9]). We want to have a closer look on the identification of $\mathrm{GL}(r)/\mathrm{O}(p,q)$ with $\mathrm{Sym}_{p,q}(\mathbb{R}^r)$ via Ψ.

To do this we choose $o = e\mathrm{O}(p,q)$ as base point and suppose that Ψ is chosen to map o to $I = I_{p,q}$. By construction Ψ is $\mathrm{GL}(r)$-equivariant. Further we identify the tangent-space $T_S \operatorname{Sym}_{p,q}(\mathbb{R}^r)$ at $S \in \operatorname{Sym}_{p,q}(\mathbb{R}^r)$ with the (ambient) vector space of symmetric matrices:

$$T_S \operatorname{Sym}_{p,q}(\mathbb{R}^r) = \operatorname{Sym}(\mathbb{R}^r) := \{A \in \operatorname{Mat}(r,\mathbb{R}) \mid A^t = A\}. \tag{5.4}$$

For $\Psi(\tilde{S}) = S$, the tangent space $T_{\tilde{S}} S(p,q)$ is canonically identified with the vector space of S-symmetric matrices:

$$T_{\tilde{S}} S(p,q) = \operatorname{sym}(S) := \{A \in \mathfrak{gl}(r) \mid A^t S = SA\}. \tag{5.5}$$

Note that $\operatorname{sym}(I_{p,q}) = \operatorname{sym}(p,q)$.

Proposition 12. *The differential of $\varphi := \Psi^{-1}$ at $S \in \operatorname{Sym}_{p,q}(\mathbb{R}^r)$ is given by*

$$\operatorname{Sym}(\mathbb{R}^r) \ni X \mapsto -\frac{1}{2} S^{-1} X \in S^{-1} \operatorname{Sym}(\mathbb{R}^r) = \operatorname{sym}(S). \tag{5.6}$$

Using this proposition we relate now the differentials

$$df_x : T_x M \to \operatorname{Sym}(\mathbb{R}^r) \tag{5.7}$$

of a map $f : M \to \operatorname{Sym}_{p,q}(\mathbb{R}^r)$ at $x \in M$ and

$$d\tilde{f}_x : T_x M \to \operatorname{sym}(f(x)) \tag{5.8}$$

of the map $\tilde{f} = \varphi \circ f : M \to S(p,q)$: $d\tilde{f}_x = d\varphi\, df_x = -\frac{1}{2} f(x)^{-1} df_x$. Analogous we have identified $\mathrm{SL}(r)/\mathrm{SO}(p,q)$ with $\operatorname{Sym}^1_{p,q}(\mathbb{R}^r)$ via Ψ^0. Let us choose $o = e\mathrm{SO}(p,q)$ as base point and suppose that Ψ^0 is chosen to map the base point o to $I = I_{p,q}$. By construction Ψ^0 is $\mathrm{SL}(r)$-equivariant. We identify the tangent space $T_S \operatorname{Sym}^1_{p,q}(\mathbb{R}^r)$ at $S \in \operatorname{Sym}^1_{p,q}(\mathbb{R}^r)$ with the (ambient) vector space of symmetric trace-free matrices:

$$T_S \operatorname{Sym}^1_{p,q}(\mathbb{R}^r) = \operatorname{Sym}_0(\mathbb{R}^r) := \{A \in \operatorname{Mat}(r,\mathbb{R}) \mid A^t = A,\ \operatorname{tr} A = 0\}. \tag{5.9}$$

For $\Psi^0(\tilde{S}) = S$, the tangent space $T_{\tilde{S}} S^1(p,q)$ is canonically identified with the vector space of S-symmetric matrices in $\mathfrak{sl}(r)$:

$$T_{\tilde{S}} S^1(p,q) = \operatorname{sym}_0(S) := \{A \in \mathfrak{sl}(r) \mid A^t S = SA\}. \tag{5.10}$$

Proposition 13. *The differential of $\varphi^0 := (\Psi^0)^{-1}$ at $S \in \operatorname{Sym}^1_{p,q}(\mathbb{R}^r)$ is given by*

$$\operatorname{Sym}_0(\mathbb{R}^r) \ni X \mapsto -\frac{1}{2} S^{-1} X \in S^{-1} \operatorname{Sym}_0(\mathbb{R}^r) = \operatorname{sym}_0(S). \tag{5.11}$$

We relate as above the differentials

$$df_x : T_x M \to \operatorname{Sym}_0(\mathbb{R}^r) \tag{5.12}$$

of a map $f : M \to \operatorname{Sym}^1_{p,q}(\mathbb{R}^r)$ at $x \in M$ and

$$d\tilde{f}_x : T_x M \to \operatorname{sym}_0(f(x)) \tag{5.13}$$

of the map $\tilde{f} = \varphi \circ f : M \to S^1(p,q)$: $d\tilde{f}_x = d\varphi\, df_x = -\frac{1}{2} f(x)^{-1} df_x$.

For a map $f : M \to \mathrm{GL}(r)$ respectively $\mathrm{SL}(r)$ we want to interpret the one-form $A = f^{-1} df$ with values in $\mathfrak{gl}(r)$ respectively $\mathfrak{sl}(r)$ as a connection form on the vector bundle $E = M \times \mathbb{R}^r$. We note, that the definition of A is the pure gauge and consequently its curvature vanishes. Thus we get:

Proposition 14.

(1) *Let $f : M \to \mathrm{GL}(r)$ be a C^∞-mapping and $A := f^{-1} df : TM \to \mathfrak{gl}(r)$. Then the curvature of A vanishes, i.e., for $X, Y \in \Gamma(TM)$*

$$Y(A_X) - X(A_Y) = A_{[X,Y]} + [A_X, A_Y]. \tag{5.14}$$

(2) *Let $f : M \to \mathrm{SL}(r)$ be a C^∞-mapping and $A := f^{-1} df : TM \to \mathfrak{sl}(r)$. Then the curvature of A vanishes, i.e., for $X, Y \in \Gamma(TM)$*

$$Y(A_X) - X(A_Y) = A_{[X,Y]} + [A_X, A_Y]. \tag{5.15}$$

In the next proposition we give the para-pluriharmonic equations for maps from a para-complex manifold to $\mathrm{GL}(r)$ respectively $\mathrm{SL}(r)$:

Proposition 15. *Let (M, τ) be a para-complex manifold $f : M \to \mathrm{GL}(r)$ or $\mathrm{SL}(r)$ a C^∞-map and A defined as in Proposition 14.*

The para-pluriharmonicity of f is equivalent to the equation

$$Y(A_X) + \frac{1}{2}[A_Y, A_X] - \tau Y(A_{\tau X}) - \frac{1}{2}[A_{\tau Y}, A_{\tau X}] = 0, \tag{5.16}$$

for all para-holomorphic vector fields $X, Y \in \Gamma(TM)$.

Proof. Again the para-pluriharmonicity of f does not depend on the connection on M. Therefore we only have to regard the pulled back Levi-Civita connection ∇ on $G = \mathrm{GL}(r)$ respectively $G = \mathrm{SL}(r)$. Let $X, Y \in \Gamma(TM)$ be para-holomorphic. To find these equations we write $df(X)$ and $df(Y)$ that are sections in f^*TG, as linear combination of left invariant vector fields $\tilde{E}_{ij}$ along f, with $\tilde{E}_{ij}(g) = gE_{ij}$, $\forall g \in G$ and a basis E_{ij}, $i, j = 1, \ldots, r$, of $\mathfrak{g} = \mathfrak{gl}(r)$ or $\mathfrak{sl}(r)$.

In this notation we have

$$df(X) = \sum_{i,j} a_{ij} \tilde{E}_{ij} \circ f = \sum_{ij} a_{ij} f E_{ij} \quad \text{and} \quad df(Y) = \sum_{i,j} b_{ij} \tilde{E}_{ij} \circ f = \sum_{ij} b_{ij} f E_{ij},$$

with functions a_{ij} and b_{ij} on M and further

$$A_X = f^{-1} df(X) = \sum_{ij} a_{ij} E_{ij} \quad \text{and} \quad A_Y = f^{-1} df(Y) = \sum_{ij} b_{ij} E_{ij}.$$

With this information we compute

$$\begin{aligned}(f^*\nabla)_Y df(X) &= (f^*\nabla)_Y \sum_{i,j} a_{ij} \tilde{E}_{ij} \circ f \\ &= \sum_{ij} Y(a_{ij}) \tilde{E}_{ij} \circ f + \sum_{ij} a_{ij} (f^*\nabla)_Y \tilde{E}_{ij} \circ f\end{aligned}$$

$$
\begin{aligned}
&= \sum_{ij} Y(a_{ij})\tilde{E}_{ij} \circ f + \sum_{ij} a_{ij} \nabla_{df(Y)} \tilde{E}_{ij} \circ f \\
&= \sum_{ij} Y(a_{ij}) f E_{ij} + \sum_{abij} a_{ij} b_{ab} \underbrace{(\nabla_{\tilde{E}_{ab}} \tilde{E}_{ij}) \circ f}_{\frac{1}{2} f[E_{ab}, E_{ij}]} \\
&= f\left(Y(A_X) + \frac{1}{2}[A_Y, A_X]\right).
\end{aligned}
$$

Therefore the para-pluriharmonicity is equivalent to the equation

$$Y(A_X) + \frac{1}{2}[A_Y, A_X] - \tau Y(A_{\tau X}) - \frac{1}{2}[A_{\tau Y}, A_{\tau X}] = 0. \qquad \square$$

6. Para-tt^*-geometry and para-pluriharmonic maps

In this section we are going to state and prove the main result, i.e., the correspondence between ptt^*-bundles and para-pluriharmonic maps. Like in Section 5 one regards the mapping $A = f^{-1}\,df$ as a map $A : TM \to \mathfrak{gl}(r)$ or $\mathfrak{sl}(r)$.

6.1. The simply-connected case

Theorem 3. *Let (M, τ) be a simply-connected para-complex manifold.*

(a) *Let (E, D, S, g) be a metric ptt^*-bundle where E has rank r and M dimension n.* *The representation of the metric g in a D^θ-flat frame of E $f : M \to \mathrm{Sym}_{p,q}(\mathbb{R}^r)$ induces a para-pluriharmonic map $\tilde{f} : M \xrightarrow{f} \mathrm{Sym}_{p,q}(\mathbb{R}^r) \tilde{\to} S(p, q)$, where $S(p, q)$ carries the pseudo-Riemannian metric induced by the (bi-invariant) trace-form on $\mathrm{GL}(r)$. Moreover, for all $x \in M$ the image of $T^{1,0}M$ under the para-complex linear extension of $dL_u^{-1}\, d\tilde{f}_x : T_xM \to T_oS(p, q) = \mathrm{sym}(p, q)$ consists of commuting matrices, where $u \in \mathrm{GL}(r)$ is any element such that $f(x) = u \cdot o$ and $L_u : S(p, q) \to S(p, q)$ is the isometry induced by the left-multiplication with $u \in \mathrm{GL}(r)$.* *Let s' be another D^θ-flat frame. Then $s' = s \cdot U$ for a constant matrix U and the para-pluriharmonic map associated to S' is $f' = U^t f U$.*

(b) *Let (E, D, S, g, or) be an oriented unimodular metric ptt^*-bundle where E has rank r and M dimension n. The representation of the metric g in a D^θ-flat frame of E $f : M \to \mathrm{Sym}^1_{p,q}(\mathbb{R}^r)$ induces a para-pluriharmonic map $\tilde{f} : M \xrightarrow{f} \mathrm{Sym}^1_{p,q}(\mathbb{R}^r) \tilde{\to} S^1(p, q)$, where $S^1(p, q)$ carries the pseudo-Riemannian metric induced by the (bi-invariant) trace-form on $\mathrm{SL}(r)$. Moreover, for all $x \in M$ the image of $T^{1,0}M$ under the para-complex linear extension of $dL_u^{-1}\, d\tilde{f}_x : T_xM \to T_oS^1(p, q) = \mathrm{sym}_0(p, q)$ consists of commuting matrices, where $u \in \mathrm{SL}(r)$ is any element such that $f(x) = u \cdot o$ and $L_u : S^1(p, q) \to S^1(p, q)$ is the isometry induced by the left-multiplication with $u \in \mathrm{SL}(r)$.* *Let s' be another D^θ-flat frame. Then $s' = s \cdot U$ for a constant matrix U and the para-pluriharmonic map associated to S' is $f' = U^t f U$.*

Remark 5 (*see also* [4]). Before proving the theorem we make some remarks on the condition on $d\tilde{f}$. Let $x \in M$ and $\tilde{f}(x) = uo$. If $d\tilde{f}(T_x^{1,0}M)$ consist of commuting matrices, then $dL_u^{-1}d\tilde{f}(T_x^{1,0}M)$ is commutative, too. This follows from the fact, that

$$dL_u : T_oS(p,q) \to T_{uo}S(p,q) = T_{\tilde{f}(x)}S(p,q)$$

equals

$$Ad_u : \mathrm{sym}(p,q) = \mathrm{sym}(I_{p,q}) \to \mathrm{sym}(u \cdot I_{p,q}) = \mathrm{sym}(\tilde{f}),$$

which preserves the Lie-bracket. An analogous remark holds for the second part of the theorem. In the sequel, we will say shortly, that the image of $d\tilde{f}_x$ is Abelian.

Proof. Using Remark 3.1) it suffices to prove the case $\theta = 0$.

(a) Let $s := (s_1, \ldots, s_r)$ be a D^0-flat frame of E (i.e., $Ds = -Ss$), f the matrix $g(s_k, s_l)$ and further S^s the matrix-valued one-form representing S in the frame s. For $X \in \Gamma(TM)$ we get:

$$\begin{aligned} X(f) &= Xg(s,s) = g(D_Xs, s) + g(s, D_Xs) \\ &= -\big(g(S_Xs, s) + g(s, S_Xs)\big) \\ &= -2g(S_Xs, s) = -2f \cdot S^s(X) = -2f \cdot S^s_X. \end{aligned}$$

Consequently $A_X = -2S^s_X$. We now prove the para-pluriharmonicity using

$$d^DS(X,Y) = D_X(S_Y) - D_Y(S_X) - S_{[X,Y]} = 0, \tag{6.1}$$

$$d^DS_\tau(X,Y) = D_X(S_{\tau Y}) - D_Y(S_{\tau X}) - S_{\tau[X,Y]} = 0. \tag{6.2}$$

Eq. (6.2) implies

$$\begin{aligned} 0 = d^DS_\tau(\tau X, Y) &= D_{\tau X}(S_{\tau Y}) - \underbrace{D_Y(S_X)}_{\overset{(6.1)}{=} D_X(S_Y) - S_{[X,Y]}} - S_{\tau[\tau X,Y]} \\ &= D_{\tau X}(S_{\tau Y}) - D_X(S_Y) + S_{[X,Y]} - S_{\tau[\tau X,Y]}. \end{aligned}$$

In local para-holomorphic coordinate fields X, Y on M we get in the frame s

$$\tau X(S^s_{\tau Y}) - X(S^s_Y) + [S^s_X, S^s_Y] - [S^s_{\tau X}, S^s_{\tau Y}] = 0.$$

Now $A = -2S^s$ gives Eq. (5.16) and hence proves the para-pluriharmonicity of f.

Using $A_X = -2S^s_X = -2d\tilde{f}(X)$, we find the property of the differential, as $S \wedge S$ is of type $(1,1)$ by the tt^*-equations, see Proposition 7.

The last statement is obvious.

(b) In this case we can take the frame s to be oriented and of volume 1, with respect to the canonical D^θ-parallel-metric volume ν. Therefore f takes values in $\mathrm{Sym}^1_{p,q}(\mathbb{R}^r)$ and part (a) shows the rest. □

Theorem 4. *Let (M,τ) be a simply-connected para-complex manifold and put $E = M \times \mathbb{R}^r$.*

(a) *Then a para-pluriharmonic map $\tilde{f} : M \to S(p,q)$ gives rise to a para-pluriharmonic map $f : M \xrightarrow{\tilde{f}} S(p,q) \tilde{\to} \mathrm{Sym}_{p,q}(\mathbb{R}^r) \subset \mathrm{GL}(r)$.*

If for all $x \in M$ $d^c \tilde{f}_x$ takes $T^{1,0}M$ to an Abelian sub-algebra in $\mathrm{sym}(f(x)) \otimes C$ (see Theorem 3), then the map f induces a metric ptt^*-bundle $(E, D = \partial - S, S = d\tilde{f}, g = \langle f\cdot,\cdot\rangle_{\mathbb{R}^r})$ on M where ∂ is the canonical flat connection on E.

(b) *Then a para-pluriharmonic map $\tilde{f}: M \to S^1(p,q)$ gives rise to a para-pluriharmonic map $f: M \xrightarrow{\tilde{f}} S^1(p,q) \tilde{\to} \mathrm{Sym}^1_{p,q}(\mathbb{R}^r) \subset \mathrm{SL}(r)$.*

If for all $x \in M$ $d^c \tilde{f}_x$ takes $T^{1,0}M$ to an Abelian sub-algebra in $\mathrm{sym}_0(f(x)) \otimes C$, then the map f induces an oriented unimodular metric ptt^*-bundle $(E, D = \partial - S, S = d\tilde{f}, g = \langle f\cdot,\cdot\rangle_{\mathbb{R}^r}, or)$ on M where ∂ is the canonical flat connection and or the canonical orientation on E.

Remark 6. We observe, that for para-complex curves $M = \Sigma$ the condition on the differential holds, since $T^{1,0}\Sigma$ is one-dimensional over C.

Proof. Let $\tilde{f}: M \to S(p,q)$ be a para-pluriharmonic map. Then by Proposition 11 we know, that $f: M \tilde{\to} \mathrm{Sym}_{p,q}(\mathbb{R}) \subset \mathrm{GL}(r)$ is para-pluriharmonic.

Since $E = M \times \mathbb{R}^r$, we can regard sections of E as r-tuples of $C^\infty(M,\mathbb{R})$-functions.

In the spirit of Section 5 we regard the one form $A = -2\,d\tilde{f} = f^{-1}df$ with values in $\mathfrak{gl}(\mathfrak{r})$ respectively $\mathfrak{sl}(\mathfrak{r})$ as a connection on E. We remind, that the curvature of this connection vanishes (Proposition 14).

(a) First, we check the conditions on the metric:

Lemma 2. *The connection D is compatible with the metric g and S is symmetric with respect to g.*

Proof. This is a direct computation with $X \in \Gamma(TM)$ and $v, w \in \Gamma(E)$ using the relations $(*)$ $S = -\frac{1}{2} f^{-1} df$, $(**)$ $df_x : T_xM \to T_{f(x)}\mathrm{Sym}_{p,q}(\mathbb{R}^r) = \mathrm{Sym}(\mathbb{R}^r)$ (compare Remark 4) and $g = \langle f\cdot,\cdot\rangle_{\mathbb{R}^r} = \langle\cdot, f\cdot\rangle_{\mathbb{R}^r}$ which follows from $f: M \to \mathrm{Sym}_{p,q}(\mathbb{R}^r)$:

$$\begin{aligned}
X\big(g(v,w)\big) &= X\big(\langle fv, w\rangle_{\mathbb{R}^r}\big) = \big\langle X(f)v, w\big\rangle_{\mathbb{R}^r} + \big\langle f(\partial_X v), w\big\rangle_{\mathbb{R}^r} + \langle fv, \partial_X w\rangle_{\mathbb{R}^r} \\
&\overset{(**)}{=} \frac{1}{2}\big\langle X(f)v, w\big\rangle_{\mathbb{R}^r} + \frac{1}{2}\big\langle v, X(f)w\big\rangle_{\mathbb{R}^r} + \big\langle f(\partial_X v), w\big\rangle_{\mathbb{R}^r} + \langle fv, \partial_X w\rangle_{\mathbb{R}^r} \\
&= \frac{1}{2}\big\langle f\cdot f^{-1}\big(X(f)\big)v, w\big\rangle_{\mathbb{R}^r} + \frac{1}{2}\big\langle v, f\cdot f^{-1}\big(X(f)\big)w\big\rangle_{\mathbb{R}^r} \\
&\quad + \big\langle f\partial_X v, w\big\rangle_{\mathbb{R}^r} + \langle fv, \partial_X w\rangle_{\mathbb{R}^r} \\
&\overset{(*),(**)}{=} g(X.v - S_X v, w) + g(v, X.w - S_X w) = g(D_X v, w) + g(v, D_X w).
\end{aligned}$$

For $x \in M$ $d\tilde{f}_x$ takes by Remark 4 values in $\mathrm{sym}(f(x))$. This shows that $S = d\tilde{f}$ is symmetric with respect to $g = \langle f\cdot,\cdot\rangle_{\mathbb{R}^r}$. $\square$

To finish the proof, we have to check the ptt^*-equations. The second ptt^*-equation

$$[S_X, S_Y] = -[S_{\tau X}, S_{\tau Y}] \tag{6.3}$$

for S follows from the assumption that the image of $T^{1,0}M$ under $d^c\tilde{f}$ is Abelian. In fact, this is equivalent to $[d\tilde{f}(\tau X), d\tilde{f}(\tau Y)] = -[d\tilde{f}(X), d\tilde{f}(Y)]$, $\forall X, Y \in TM$.

$$\begin{aligned}
d^D S(X,Y) &= [D_X, S_Y] - [D_Y, S_X] - S_{[X,Y]} \\
&= \partial_X(S_Y) - \partial_Y(S_X) - 2[S_X, S_Y] - S_{[X,Y]} = 0
\end{aligned}$$

is equivalent to the vanishing of the curvature of $A = -2S$ interpreted as a connection on E (see Proposition 14).

Finally one has for para-holomorphic coordinate fields $X, Y \in \Gamma(TM)$

$$\begin{aligned} d^D S_\tau(\tau X, Y) &= [D_{\tau X}, S_{\tau Y}] - [D_Y, S_X] = [\partial_{\tau X} - S_{\tau X}, S_{\tau Y}] - [\partial_Y - S_Y, S_X] \\ &= \partial_{\tau X}(S_{\tau Y}) - \partial_Y(S_X) - [S_{\tau X}, S_{\tau Y}] - [S_X, S_Y] \\ &\overset{(6.3)}{=} \frac{1}{2}\big(\partial_Y(A_X) - \partial_{\tau X}(A_{\tau Y})\big) \\ &\overset{(5.14)}{=} \frac{1}{2}\big(\partial_X(A_Y) + [A_X, A_Y] - \partial_{\tau X}(A_{\tau Y})\big) \\ &\overset{(6.3)}{=} \frac{1}{2}\left(\partial_X(A_Y) + \frac{1}{2}[A_X, A_Y] - \partial_{\tau X}(A_{\tau Y}) - \frac{1}{2}[A_{\tau X}, A_{\tau Y}]\right) \overset{(5.16)}{=} 0. \end{aligned}$$

This shows the vanishing of the tensor $d^D S_\tau$.

It remains to show the curvature equation for D. We observe, that $D + S = \partial$ and is consequently flat, to find

$$0 = R^{D+S}_{X,Y} = R^D_{X,Y} + d^D S(X, Y) + [S_X, S_Y] \overset{d^D S = 0}{=} R^D_{X,Y} + [S_X, S_Y].$$

(b) With the same proof as in part a) we get a metric ptt^*-bundle with orientation.

It remains to check the condition on the trace of S. This property is clear, since in this case $d\tilde{f}_x$ takes values in $\mathrm{sym}_0(f(x))$ for all $x \in M$. $\square$

In the situation of Theorem 4 the two constructions are inverse in the following sense:

Proposition 16.

(1) *Let (E, D, S, g) be a metric ptt^*-bundle over a para-complex manifold (M, τ) and let $\tilde{f}$ be the associated para-pluriharmonic map constructed to a D^θ-flat frame s in Theorem 3. Then the image of $d\tilde{f}$ is Abelian and the metric ptt^*-bundle $(M \times \mathbb{R}^r, \tilde{D} = \partial - \tilde{S}, \tilde{S} = d\tilde{f}, \tilde{g})$ associated to $\tilde{f}$ in Theorem 4 is the representation of (E, D, S, g) in the frame s.*

(2) *Given a para-pluriharmonic map $\tilde{f}$ from a para-complex manifold (M, τ) to $S(p, q)$ such that the image of $d\tilde{f}$ is Abelian, one obtains via Theorem 4 a metric ptt^*-bundle $(M \times \mathbb{R}^r, D, S, g)$.*
The para-pluriharmonic map associated to this metric ptt^*-bundle is conjugated to the map $\tilde{f}$ by a constant matrix in $\mathrm{GL}(r)$.

(3) *Let (E, D, S, g, or) be an oriented unimodular metric ptt^*-bundle over a para-complex manifold (M, τ). Let $\tilde{f}$ be the associated para-pluriharmonic map constructed to a D^θ-flat frame s of canonical volume one in Theorem 3. Then the image of $d\tilde{f}$ is Abelian and the oriented unimodular metric ptt^*-bundle $(M \times \mathbb{R}^r, \tilde{D} = \partial - \tilde{S}, \tilde{S} = d\tilde{f}, \tilde{g}, or')$ associated to $\tilde{f}$ of Theorem 4 is its representation in the frame s.*

(4) *Given a para-pluriharmonic map $\tilde{f}$ from a para-complex manifold (M, τ) to $S^1(p, q)$ such that the image of $d\tilde{f}$ is Abelian, one obtains via Theorem 4 an oriented unimodular metric ptt^*-bundle $(M \times \mathbb{R}^r, D, S, g, or)$.*
The para-pluriharmonic map associated to this oriented unimodular metric ptt^*-bundle is conjugated to the map $\tilde{f}$ by a constant matrix in $\mathrm{SL}(r)$.

Proof. Using again Remark 3.1) we can set $\theta = 0$.

(1) The maps f, $\tilde{f}$ and the metric $\tilde{g} = \langle f \cdot, \cdot \rangle_{\mathbb{R}^r}$ express the metric g in the frame s. In the computations of Theorem 3 and with Theorem 4 one finds $2\tilde{S} = -A = -f^{-1}\,df = 2S^s$. From $0 = D^0 s = Ds + Ss$ we obtain that the connection D in the frame s is just $\partial - S^s = \partial + \frac{A}{2} = \partial - \tilde{S} = \tilde{D}$.
(2) To find the para-pluriharmonic map associated to $(M \times \mathbb{R}^r, D, S, g)$ we have to express the metric g in a D^0-flat frame s. But $D^0 = D + S = \partial - S + S = \partial$. Hence we can take s as the standard-basis of $\mathbb{R}^r$ and we get f. Every other basis gives a conjugated result.
(3) As part (1).
(4) As part (2). $\square$

6.2. The general case

In this subsection we are going to transfer the results in the simply-connected case to manifolds with non-trivial fundamental group.

Definition 12. Let $p : \tilde{M} \to M$ the universal cover of a para-complex manifold (M, τ) with the pulled back para-complex structure.

Let (E, D, S) be a ptt^*-bundle, then we define the pulled back ptt^*-bundle of (E, D, S) to be given by (p^*E, p^*D, p^*S).

Let (E, D, S, g) be a metric ptt^*-bundle, then we define the pulled back metric ptt^*-bundle of (E, D, S, g) to be given by (p^*E, p^*D, p^*S, p^*g).

Finally, let (E, D, S, g, or) be an oriented unimodular metric ptt^*-bundle, then we define the pulled back oriented unimodular metric ptt^*-bundle of (E, D, S, g, or) to be given by $(p^*E, p^*D, p^*S, p^*g, p^*or)$.

Remark 7. The pulled back ptt^*-bundles, metric ptt^*-bundles, oriented and unimodular metric ptt^*-bundles are ptt^*-bundles, metric ptt^*-bundles, oriented and oriented unimodular metric ptt^*-bundles respectively, as one easily checks. This motivates the above definition.

Theorem 5. *Let (M, τ) be a para-complex manifold.*

(a) *Let (E, D, S, g) be a metric ptt^*-bundle where E has rank r and M dimension n and (p^*E, p^*D, p^*S, p^*g) the corresponding pulled-back metric ptt^*-bundle on the universal cover $\tilde{M}$ of M.*
Denote $f^* : \tilde{M} \to S(p,q)$ the para-pluriharmonic map obtained from Theorem 3 in the p^*D^θ-flat frame p^*s, where s is a D^θ-flat frame and $f : M \to S(p,q)$ the map obtained from the representation of g in the frame s. Then f^* is a $\pi_1(M)$-equivariant map (Here equivariant means by the left-action on M and via the holonomy on $S(p,q)$.) and the lift p^*f of f. In other words f is a twisted para-pluriharmonic map.
(b) *Let (E, D, S, g, or) be an oriented unimodular metric ptt^*-bundle where E has rank r and M dimension n and $(p^*E, p^*D, p^*S, p^*g, p^*or)$ the corresponding pulled back metric ptt^*-bundle on the universal cover $\tilde{M}$ of M.*
Denote $f^* : \tilde{M} \to S^1(p,q)$ the para-pluriharmonic map obtained from Theorem 3 in the p^*D^θ-flat frame p^*s, where s is a D^θ-flat frame and $f : M \to S^1(p,q)$ the map obtained from the represen-

tation of g in the frame s. Then f^* is a $\pi_1(M)$-equivariant map (Here equivariant means by the left-action on M and via the holonomy on $S^1(p,q)$.) and f^* is the lift p^*f of f. In other words f is a twisted para-pluriharmonic map.

Proof. The equivariance follows, since we have pulled back all structures. If s is D^θ-flat, p^*s is p^*D^θ-flat, too.

The map f^* at $\tilde{x} \in \tilde{M}$ with $p(\tilde{x}) = x$ is given by

$$f^*(\tilde{x}) = p^*g(p^*s, p^*s)(x) = g_{p(\tilde{x})}\big(s \circ p(\tilde{x}), s \circ p(\tilde{x})\big) = f(x) = f \circ p(\tilde{x}) = p^*f(x). \qquad \square$$

Theorem 6. *Let (M,τ) be a para-complex manifold, $p : \tilde{M} \to M$ its universal covering with the pulled back para-complex structure, also called τ. Set $E = \tilde{M} \times \mathbb{R}^r$.*

(a) *Let $\tilde{f}^* : \tilde{M} \to S(p,q)$ be a para-pluriharmonic map, which is equivariant with respect to a representation $\rho : \pi_1(M) \to \mathrm{GL}(r)$ and $f^* : \tilde{M} \to \mathrm{Sym}_{p,q}(\mathbb{R}^r)$ the corresponding map, such that for all $x \in M$ the image of $d\tilde{f}^*{}_x$ is Abelian in $\mathrm{sym}(f^*(x)) \otimes C$.*
Then $\tilde{f}^*$ induces by Theorem 4 a metric ptt^*-bundle $(E, D = \partial - S, S = d\tilde{f}^*, g = \langle f^* \cdot, \cdot\rangle_{\mathbb{R}^r})$ on $\tilde{M}$ where ∂ is the canonical flat connection on E. This metric ptt^*-bundle induces a metric ptt^*-bundle $(F, D = \partial - T, T, h)$ on M, such that $(E, D = \partial - S, S = d\tilde{f}^*, g = \langle f^* \cdot, \cdot\rangle_{\mathbb{R}^r})$ is its pull back.

(b) *Let $\tilde{f}^* : \tilde{M} \to S^1(p,q)$ be a para-pluriharmonic map, which is equivariant with respect to a representation $\rho : \pi_1(M) \to \mathrm{SL}(r)$ and $f^* : \tilde{M} \to \mathrm{Sym}^1_{p,q}(\mathbb{R}^r)$ the corresponding map, such that for all $x \in M$ the image of $d\tilde{f}^*{}_x$ is Abelian in $\mathrm{sym}_0(f^*(x)) \otimes C$.*
Then $\tilde{f}^*$ induces by Theorem 4 an oriented unimodular metric ptt^*-bundle $(E, D = \partial - S, S = d\tilde{f}^*, g = \langle f^* \cdot, \cdot\rangle_{\mathbb{R}^r}, or)$ on $\tilde{M}$ where ∂ is the canonical flat connection on E. This oriented unimodular metric ptt^*-bundle induces an oriented unimodular metric ptt^*-bundle $(F, D = \partial - T, T, h, or')$ on M, such that $(E, D = \partial - S, S = d\tilde{f}^*, g = \langle f^* \cdot, \cdot\rangle_{\mathbb{R}^r}, or)$ is its pull back.

Proof. (a) We want to regard the action of $\pi_1(M)$ on E, given by

$$(\gamma, m, v) \in \pi_1(M) \times E \mapsto \big(\gamma.m, \rho(\gamma)v\big) =: \gamma.(m,v) \in E \tag{6.4}$$

which induces the action

$$(\gamma, m, A) \in \pi_1(M) \times \mathrm{End}(E) \mapsto \big(\gamma.m, \rho(\gamma)A\rho(\gamma)^{-1}\big) =: \gamma.(m,A) \in \mathrm{End}(E) \tag{6.5}$$

of $\pi_1(M)$ on $\mathrm{End}(E)$. The quotient of E by the action of $\pi_1(M)$ gives a vector-bundle $F \to M$ over M.

The equivariance of the map $\tilde{f}^* : \tilde{M} \to S(p,q)$ means for $m \in \tilde{M}$:

$$\tilde{f}^*(\gamma.m) = \rho(\gamma)\tilde{f}^*(m)\rho(\gamma)^{-1}, \tag{6.6}$$

which implies for $X \in T_m\tilde{M}$, $m \in \tilde{M}$

$$d\tilde{f}^*_{\gamma.m}(d\gamma X) = \rho(\gamma) d\tilde{f}^*_m(X)\rho(\gamma)^{-1}. \tag{6.7}$$

Eq. (6.6) is the equivariance of g and Eq. (6.7) the equivariance of S. Hence they descend to a metric h on F and an endomorphism field T on F, which is h-symmetric. Since ∂ is $\pi_1(M)$-invariant, it defines connection on F and since S is equivariant $D = \partial - T$ defines connection on F which preserves h. With the same argument the family $D^\theta = D + \cosh(\theta)T + \sinh(\theta)T_\tau$ defines a family of connections on F which is flat. Hence $(F, D = \partial - T, T, h)$ is a ptt^*-bundle on F over M.

(b) One gets the data $(F, D = \partial - T, T, h)$ as in part (a). The orientation is given by the orientation of $E = \tilde{M} \times \mathbb{R}^r$, since ρ take values in $\mathrm{SL}(r)$. □

7. The para-pluriharmonic map in the special para-Kähler case

7.1. The extrinsic description of special para-Kähler manifolds

In this section we are going to recall the extrinsic description of special para-Kähler manifolds given in [3].

First we have to introduce a canonical non-degenerated exact C-valued two-form Ω of type $(2,0)$ on the cotangent bundle $N = T^*M$ of an arbitrary para-complex manifold (M,τ), which is para-holomorphic, i.e., it is a para-holomorphic section of the para-holomorphic vector-bundle $\Lambda^{2,0}T^*N$. Its explicit form is given by the following consideration: We take local para-holomorphic coordinates $(z^1,\dots,z^n)$ on an open subset $U \subset M^n$. Any point of $T_p^*M \cong \mathrm{Hom}(T_p^*M, \mathbb{R}) \cong \mathrm{Hom}_C(T_p^*M, C)$, $p \in U$, where $\mathrm{Hom}_C(T_p^*M, C)$ are the homomorphisms from the para-complex vector space (T_p^*M, τ_p) to C, can be expressed as $\sum w_i\, dz^i_{|p}$. The coordinates z^i and w_i can be regarded as local para-holomorphic coordinates of the bundle $T^*M_{|U}$. The coordinates w_i induce linear para-holomorphic coordinates on each fiber T_p^*M for $p \in U$. In these coordinates the two form Ω is given by

$$\Omega = \sum dz^i \wedge dw_i = -d\Big(\sum w_i\, dz^i\Big).$$

We observe, that $\sum w_i\, dz^i$ does not depend on the choice of coordinates and hence Ω does not depend on the choice of coordinates, too. The form Ω will be called the *symplectic form* of T^*M.

In the following, we denote by V the para-holomorphic vector-space $T^*C^n = C^{2n}$, endowed with its standard para-complex structure τ_V, its symplectic form Ω and the para-complex conjugation $\bar{\cdot}: V \to V$, $v \mapsto \bar{v}$ with fixed point set $T^*\mathbb{R}^n \cong \mathbb{R}^{2n}$. On this space we take a system of para-holomorphic linear coordinates (z^i, w_i) which are real-valued on $T^*\mathbb{R}^n$. The algebraic data (Ω, τ) defines a para-Hermitian scalar product on V via

$$g_V(v,w) = \mathrm{Re}\,\gamma(v,w) = \mathrm{Re}\big(e\Omega(v,\bar{w})\big), \quad \forall v,w \in V \text{ with } \gamma(v,w) = e\Omega(v,\bar{w})$$

and (V, τ_V, g_V) is a flat para-Kähler manifold, whose para-Kähler form is given by

$$\omega_V(v,w) := g_V(\tau_V v, w) = \mathrm{Im}\big(e\Omega(v,\bar{w})\big), \quad \forall v,w \in V.$$

Let (M,τ) be a para-complex manifold. We call a para-holomorphic immersion $\phi: M \to V$ *para-Kählerian* if $g = \phi^* g_V$ is non-degenerate and *Lagrangian* if $\phi^*\Omega = 0$. Any para-Kählerian immersion $\phi: M \to V$ induces on M the structure of a para-Kähler manifold (M,τ,g) with para-Kähler form $\omega(\cdot,\cdot) = g(\tau\cdot,\cdot) = \phi^*\omega_V$. For a para-Kählerian Lagrangian immersion the para-Kähler form $\omega = g(\tau\cdot,\cdot)$ of M is given by

$$\omega = 2\sum d\tilde{x}^i \wedge d\tilde{y}_i,$$

where $\tilde{x}^i = \mathrm{Re}(\phi^* z^i)$ and $\tilde{y}^i = \mathrm{Re}(\phi^* w^i)$. Additionally, a para-Kählerian Lagrangian immersion $\phi : M \to V$ induces a canonical flat torsion-free connection ∇ on M which is characterized by the condition, that $\nabla(\mathrm{Re}\,\phi^*\,df) = 0$ for all complex affine functions f on V.

With these informations we now can give the extrinsic description of para-Kähler manifolds:

Theorem 7 [3]. *Let $\phi : M \to V$ be a para-Kählerian immersion with induced geometric data (τ, g, ∇). Then (M, τ, g, ∇) is a special para-Kähler manifold. Conversely, any simply-connected special para-Kähler manifold (M, τ, g, ∇) admits a para-Kählerian Lagrangian immersion inducing the special geometric data (τ, g, ∇) on M. The para-Kählerian Lagrangian immersion ϕ is unique up to an affine linear transformation of V whose linear part belongs to the group* $\mathrm{Aut}(V, \Omega, \bar{\cdot}) = \mathrm{Aut}_{\mathbb{R}}(V, \tau_V, \Omega, \bar{\cdot}) = \mathrm{Sp}(\mathbb{R}^{2n})$.

7.2. The Gauß maps of a special para-Kähler manifold

Now we are going to introduce the Gauß maps of a special para-Kähler manifold, which are the para-complex analogue of the Gauß maps introduced in [4].

Let (M, τ, g, ∇) be a special para-Kähler manifold of para-complex dimension n. Consequently the metric g has signature (n, n). Let $(\tilde{M}, \tau, g, \nabla)$ be the universal cover of M with the pull-back special para-Kähler structure, which we denote again by (τ, g, ∇). According to Theorem 7, there exists a (para-holomorphic) Kählerian Lagrangian immersion $\Phi : \tilde{M} \to V = C^{2n} = T^*C^n$, which is unique up to an affine transformation of V with linear part in $\mathrm{Aut}(V, \Omega, \bar{\cdot}) = \mathrm{Sp}(\mathbb{R}^{2n})$. We consider the *dual Gauß map* of ϕ

$$L : \tilde{M} \to \mathrm{Gr}_0^n(C^{2n}), \quad p \mapsto L(p) := T_{\phi(p)}\tilde{M} := d\phi_p T_p \tilde{M} \subset V$$

into the Grassmannian $\mathrm{Gr}_0^n(C^{2n})$ of para-complex Lagrangian subspaces $W \subset V$ of signature (n, n), i.e., $g_V = \mathrm{Re}\,\gamma$ restricted to W has signature (n, n). The map $L : \tilde{M} \to \mathrm{Gr}_0^n(C^{2n})$ is in fact the dual of the *Gauß map*

$$L^\perp : \tilde{M} \to \mathrm{Gr}_0^n(C^{2n}), \quad p \mapsto L(p)^\perp = \bar{L}(p) \cong L(p)^*.$$

With $L(p)^\perp$ we mean the γ-orthogonal complement of $L(p)$ and the isomorphism $L(p)^\perp \cong L(p)^*$ is induced by the symplectic form Ω on $V = L(p) \oplus \bar{L}(p)$. The structure of a para-complex manifold on $\mathrm{Gr}_0^n(C^{2n})$ is introduced in the next section.

Proposition 17.

(i) *The dual Gauß map $L : \tilde{M} \to \mathrm{Gr}_0^n(C^{2n})$ is para-holomorphic.*
(ii) *The Gauß map $L^\perp : \tilde{M} \to \mathrm{Gr}_0^n(C^{2n})$ is anti-para-holomorphic.*

Proof. The para-holomorphicity follows from that of ϕ and part (ii) follows from $L^\perp = \bar{L} : p \mapsto \bar{L}(p)$. $\square$

The Gauß maps L and $L^\perp$ induce Gauß maps

$$L : M \to \Gamma \setminus \mathrm{Gr}_0^n(C^{2n}),$$

$$L^{\perp} : M \to \Gamma \setminus \mathrm{Gr}_0^n(C^{2n})$$

in the quotient of the Grassmannian by the holonomy group $\Gamma \subset \mathrm{Hol}(\nabla) \subset \mathrm{Sp}(\mathbb{R}^{2n})$ of the flat symplectic connection ∇. This yields the

Corollary 3.

(i) *The dual Gauß map* $L_M : M \to \Gamma \setminus \mathrm{Gr}_0^n(C^{2n})$ *is para-holomorphic.*
(ii) *The Gauß map* $L_M^{\perp} : M \to \Gamma \setminus \mathrm{Gr}_0^n(C^{2n})$ *is anti-para-holomorphic.*

7.3. Coordinates on the para-complex Lagrangian Grassmannian

In this section we shall introduce a local model of the Grassmannian $\mathrm{Gr}_0^n(C^{2n})$ of para-complex Lagrangian subspaces $W \subset V$ of signature (n,n), i.e., such that $g_V = \mathrm{Re}\,\gamma$ restricted to W has signature (n,n).

This model is a para-complex pseudo-Riemannian analog to the Siegel upper half-space

$$\mathrm{Sym}^+(\mathbb{C}^n) := \{A \in \mathrm{Mat}(n,\mathbb{C}) \mid A^t = A \text{ and } \mathrm{Im}\,A \text{ is positive definite}\}. \tag{7.1}$$

The real symplectic group $\mathrm{Sp}(\mathbb{R}^{2n})$ acts transitively on $\mathrm{Gr}_0^n(C^{2n})$ and we have the following identification: $\mathrm{Gr}_0^n(C^{2n}) \cong \mathrm{Sp}(\mathbb{R}^{2n})/U^{\pi}(C^n)$, where $U^{\pi}(C^n)$ is the stabilizer of

$$W_o = \mathrm{span}_C \left\{ \frac{\partial}{\partial z^1} + e\frac{\partial}{\partial w_1}, \ldots, \frac{\partial}{\partial z^n} + e\frac{\partial}{\partial w_n} \right\} \tag{7.2}$$

and $U^{\pi}(C^n)$ is defined as the group

$$U^{\pi}(C^n) = \mathrm{Aut}(C^n, \tau_{C^n}, g_{C^n}) = \{L \in \mathrm{GL}(\mathbb{R}^{2n}) \mid \text{ s.t. } L^*g = g \text{ and } [L, \tau_{C^n}] = 0\}. \tag{7.3}$$

Given a point $W \in \mathrm{Gr}_0^n(C^{2n})$ we claim, that $V = T^*C^n$ decomposes into the direct sum

$$V = W \oplus \bar{W}. \tag{7.4}$$

Let $\gamma^W = \gamma_{|W}$, $\omega^W = (\omega_V)_{|W}$ and $g^W = (g_V)_{|W}$. Then the non-degeneracity of γ^W, g^W and ω^W are equivalent. One sees from the definition of γ^W that it is non-degenerated if and only if $W \cap \bar{W} = \{0\}$. Further it is $\dim_{\mathbb{R}}(W) = \dim_{\mathbb{R}}(\bar{W}) = \frac{\dim_{\mathbb{R}}(V)}{2}$, where the last equation follows since W is Lagrangian. This proves the claim.

One computes easily $\gamma(\bar{v}, \bar{w}) = -\gamma(w, v)$, $\forall v, w \in W$. Hence $g^{\bar{W}}$ has signature (n,n), since g^W has signature (n,n). Since $\gamma = e\Omega(\cdot, \bar{\cdot})$ and W is Lagrangian, it follows that the decomposition (7.4) is γ-orthogonal. Using the isomorphism induced by the symplectic form Ω on $V = W \oplus \bar{W}$ yields an isomorphism of $W^{\perp} = \bar{W} \cong W^*$ where $\cdot^{\perp}$ is the orthogonal complement taken with respect to γ.

We now construct para-holomorphic coordinates for the para-complex manifold $\mathrm{Gr}_0^n(C^{2n})$ in a Zariski open neighborhood of a point W_0 of the Grassmannian represented by a Lagrangian subspace $W_0 \subset V$ of signature (n,n). Using the transitive action of the group $\mathrm{Sp}(\mathbb{R}^{2n})$ on $\mathrm{Gr}_0^n(C^{2n})$ we may assume $W_0 = W_o$, see (7.2). Let $U_0 \subset \mathrm{Gr}_0^n(C^{2n})$ be the open subset consisting of $W \in \mathrm{Gr}_0^n(C^{2n})$ such that the projection

$$\pi_{(z)} : V = T^*C^n = C^n \oplus (C^n)^* \to C^n \tag{7.5}$$

onto the first summand (z-space) induces an isomorphism

$$\pi_{(z)|W} : W \tilde{\to} C^n. \tag{7.6}$$

Observe, that $U_0 \subset \mathrm{Gr}_0^n(C^{2n})$ is an open neighborhood of the base point W_o. For elements $W \in U_0$ we can express w_i as a function of $z = (z^1, \dots, z^n)$. In fact,

$$w_i = \sum C_{ij} z^j \tag{7.7}$$

where

$$C_{ij} \in \mathrm{Sym}_{n,n}(C^n) = \{A \in \mathrm{Mat}(n, C) \mid A^t = A \text{ and } \mathrm{Im}(A) \text{ has signature } (n,n)\}. \tag{7.8}$$

Proposition 18. *The map*

$$C : U_0 \to \mathrm{Sym}_{n,n}(C^n), \quad W \mapsto C(W) := (C_{ij}) \tag{7.9}$$

is a local para-holomorphic chart for the Grassmannian $\mathrm{Gr}_0^n(C^{2n})$.

We now describe the dual Gauß map L in local para-holomorphic coordinates of $p_o \in \tilde{M}$ and $L(p_0) \in \mathrm{Gr}_0^n(C^{2n})$. Utilizing a transformation of $\mathrm{Sp}(\mathbb{R}^{2n})$, if necessary, we can assume $L(p_0) \in U_0$. We put $U := L^{-1}(U_0)$. The set $U \subset \tilde{M}$ is an open neighborhood of p_0.

Let $\phi : \tilde{M} \to T^*C^n$ be the para-Kählerian Lagrangian immersion. It defines a system of local (special) para-holomorphic coordinates

$$\varphi := \pi_{(z)} \circ \phi_{|U} : U \tilde{\to} U' \subset C^n, \quad p \mapsto \left(z^1(\phi(p)), \dots, z^n(\phi(p))\right) \tag{7.10}$$

and we have the following commutative diagram

$$\begin{tikzcd} U \arrow[r, "L"] \arrow[d, "\varphi"'] & U_0 \arrow[d, "C"] \\ U' \arrow[r, "L_U"] & \mathrm{Sym}_{n,n}(C^n), \end{tikzcd} \tag{7.11}$$

where the vertical arrows are para-holomorphic diffeomorphisms and L_U at $z = (z^1, \dots, z^n)$ is given by

$$L_U(z) = \left(F_{ij}(z)\right) := \left(\frac{\partial^2 F(z)}{\partial z^i \partial z^j}\right). \tag{7.12}$$

Here $F(z)$ is a para-holomorphic function on $U' \subset C^n$, called prepotential (see [3]), which is up to a constant determined by the equations

$$w_j\left(\phi(p)\right) = \left.\frac{\partial F}{\partial z^j}\right|_{z(\phi(p))}. \tag{7.13}$$

Summarizing, we obtain the proposition

Proposition 19. *The dual Gauß map L has the following coordinate expression*

$$L_U = C \circ L \circ \varphi^{-1} = (F_{ij}), \tag{7.14}$$

where $\varphi : U \to C^n$ is the (special) para-holomorphic chart of $\tilde{M}$ associated to the para-Kählerian Lagrangian immersion ϕ, see (7.10), and $C : U_0 \to \mathrm{Sym}_{n,n}(C^n)$ is the para-holomorphic chart of $\mathrm{Gr}_0^n(C^{2n})$ constructed in (7.9).

7.4. The special para-Kähler metric in an affine frame

In this section we show that the para-pluriharmonic map associated to a para-Kähler manifold coincides with the dual Gauß map.

As above, let (M,τ,g,∇) be a special para-Kähler manifold of dimension $n=\dim_C M$ and $(\tilde M,\tau,g,\nabla)$ be its universal covering. Like in Section 6 we now consider the metric g in a ∇-parallel frame. Such a frame is provided by the para-Kählerian Lagrangian immersion $\phi:\tilde M\to V$. In fact, an arbitrary point $p\in\tilde M$ has a neighborhood in which the functions $\tilde x^i:=\operatorname{Re} z^i\circ\phi$ and $\tilde y_i:=\operatorname{Re} w_i\circ\phi$, $i=1,\dots,n$ form a system of local ∇-affine coordinates. We recall that the ∇-parallel Kähler form is given by $\omega=2\sum d\tilde x^i\wedge d\tilde y_i$. Therefore the globally defined one-forms $\sqrt2\,d\tilde x^i$ and $\sqrt2\,d\tilde y_i$ constitute a ∇-parallel unimodular frame

$$(e^a)_{a=1,\dots,2n}:=\left(\sqrt2\,d\tilde x^1,\dots,\sqrt2\,d\tilde x^n,\sqrt2\,d\tilde y_1,\dots,\sqrt2\,d\tilde y_n\right) \tag{7.15}$$

of $T^*\tilde M$ with respect to the metric volume form $\nu=(-1)^{n+1}\omega^n/n!=2^n\,d\tilde x^1\wedge\cdots\wedge d\tilde y_n$. The dual frame e_a of $T\tilde M$ is also ∇-parallel and unimodular. The metric g defines a smooth map

$$G:\tilde M\to \operatorname{Sym}^1_{n,n}(\mathbb R^{2n})=\left\{A\in\operatorname{Mat}(2n,\mathbb R)\mid A^t=A,\ \det(A)=(-1)^n \text{ of signature } (n,n)\right\}$$

by

$$p\mapsto G(p)=\left(g_{ab}(p)\right):=\left(g_p(e_a,e_b)\right). \tag{7.16}$$

We call $G=g_{ab}(p)$ the *fundamental matrix* of ϕ. As before, we have the identification

$$\operatorname{Sym}^1_{n,n}(\mathbb R^{2n})=\mathrm{SL}(2n,\mathbb R)/\mathrm{SO}(n,n)$$

of $\operatorname{Sym}^1_{n,n}(\mathbb R^{2n})$ with a pseudo-Riemannian symmetric space.

The group $\mathrm{SO}(n,n)\subset\mathrm{SL}(2n,\mathbb R)$ is the stabilizer of the symmetric matrix

$$E^n_0=\operatorname{diag}(-\mathbb 1_n,\mathbb 1_n). \tag{7.17}$$

The fundamental matrix induces a map

$$G_M:M\to\Gamma\setminus\operatorname{Sym}^1_{n,n}(\mathbb R^{2n})$$

into the quotient of $\operatorname{Sym}^1_{n,n}(\mathbb R^{2n})$ by the action of the holonomy group $\Gamma=\operatorname{Hol}(\nabla)\subset\operatorname{Sp}(\mathbb R^{2n})\subset\mathrm{SL}(2n,\mathbb R)$. The target $\Gamma\setminus\operatorname{Sym}^1_{n,n}(\mathbb R^{2n})$ is a pseudo-Riemannian locally symmetric space, provided that Γ acts properly discontinuously.

Theorem 8. *The fundamental matrix $G:\tilde M\to\operatorname{Sym}^1_{n,n}(\mathbb R^{2n})=\mathrm{SL}(2n,\mathbb R)/\mathrm{SO}(n,n)$ takes values in the totally geodesic sub-manifold*

$$i:\operatorname{Gr}^n_0(C^{2n})=\operatorname{Sp}(\mathbb R^{2n})/U^\pi(C^n)\hookrightarrow\mathrm{SL}(2n,\mathbb R)/\mathrm{SO}(n,n)$$

and coincides with the dual Gauß map $L:\tilde M\to\operatorname{Gr}^n_0(C^{2n}):G=i\circ L$.

Proof. The proof follows from a geometric interpretation of the inclusion i. To any Lagrangian subspace $W\in\operatorname{Gr}^n_0(C^{2n})$ we associate the scalar product $g^W:=\operatorname{Re}\gamma_{|W}$ of signature (n,n) on $W\subset V$. The projection onto the real points

$$\mathrm{Re}: V = T^*C^n \mapsto T^*\mathbb{R}^n = \mathbb{R}^{2n}, \quad v \mapsto \mathrm{Re}\, v = \frac{1}{2}(v + \bar{v}) \tag{7.18}$$

induces an isomorphism of real vector spaces $W \tilde{\to} \mathbb{R}^{2n}$ with inverse $\psi = \psi_W$.

We claim that

$$i(W) = \psi_W^* g =: \big(g_{ab}^W\big) =: G^W. \tag{7.19}$$

To check the claim, we have to show the $\mathrm{Sp}(\mathbb{R}^{2n})$-equivariance of

$$\mathrm{Gr}_0^n(C^{2n}) \ni W \mapsto G^W \in \mathrm{Sym}^1_{n,n}(\mathbb{R}^{2n})$$

and that it maps the base point W_o, see Eq. (7.2), to E_0^n (Eq. (7.17)).

By the definition of γ we find for the basis

$$e_j^\pm := \frac{\partial}{\partial z^j} \pm e\frac{\partial}{\partial w_j} \tag{7.20}$$

of V that the only non-vanishing components of γ are $\gamma(e_j^\pm, e_j^\pm) = \mp 2$. This shows that g^{W_o} is represented by the matrix $2E_0^n$ with respect to the real basis

$$\big(e_1^+, \dots, e_n^+, ee_1^+, \dots, ee_n^+\big). \tag{7.21}$$

In order to calculate $G^{W_o} = (g_{ab}^{W_o}) = (g(\psi e_a, \psi e_b))$, we need to pass from the real basis (7.21) to the real basis (ψe_a) of W_o.

Recall that the real structure is the para-complex conjugation with respect to the coordinates (z^i, w_i). This implies that

$$\psi^{-1}(e_j^+) = \frac{\partial}{\partial x^j} = \sqrt{2}e_j, \quad \psi^{-1}(ee_j^+) = \frac{\partial}{\partial y^j} = \sqrt{2}e_{n+j}, \qquad j = 1, \dots, n, \tag{7.22}$$

$$\psi^{-1}(e_j^-) = \frac{\partial}{\partial x^j} = \sqrt{2}e_j, \quad \psi^{-1}(ee_j^-) = -\frac{\partial}{\partial y^j} = -\sqrt{2}e_{n+j}, \quad j = 1, \dots, n. \tag{7.23}$$

This shows that $G^{W_o} = E_0^n$.

It remains to show the equivariance of $W \mapsto G^W = \psi^* g$. Using the definition of the map $\psi = \psi^W: \mathbb{R}^{2n} \to W$, one easily checks that, under the action of $\Lambda \in \mathrm{Sp}(\mathbb{R}^{2n})$, ψ transforms as

$$\psi_{\Lambda W} = \Lambda \circ \Psi_W \circ \Lambda^{-1}_{|\mathbb{R}^{2n}}. \tag{7.24}$$

This implies the transformation law for G^W:

$$G^{\Lambda W} = \psi_{\Lambda W}^* g^{\Lambda W} = (\Lambda^{-1})^* \psi_W^* \Lambda^* g^{\Lambda W} = (\Lambda^{-1})^* \psi_W^* g^W = (\Lambda^{-1})^* G^W = \Lambda \cdot G^W. \tag{7.25}$$

The above claim (7.19) and the fact

$$g^{L(p)} = g_p \quad \text{and} \quad G^{L(p)} = G(p) \tag{7.26}$$

for all $p \in \tilde{M}$ imply

$$i\big(L(p)\big) = G^{L(p)} = G(p). \qquad \square \tag{7.27}$$

Corollary 4. *The fundamental matrix* $G : \tilde{M} \to \mathrm{Sym}^1_{n,n}(\mathbb{R}^{2n})$ *is para-pluriharmonic.*

Proof. In fact, $G = i \circ L$ is the composition of a para-holomorphic map $L : \tilde{M} \to \mathrm{Gr}^n_0(C^{2n})$ with the totally geodesic inclusion $\mathrm{Gr}^n_0(C^{2n}) \subset \mathrm{Sym}^1_{n,n}(\mathbb{R}^{2n})$. The composition of a para-holomorphic map with a totally geodesic one is para-pluriharmonic. $\square$

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References

[1] D.V. Alekseevsky, C. Devchand, V. Cortés, Special complex manifolds, J. Geom. Phys. 42 (2002) 85–105.
[2] S. Cecotti, C. Vafa, Topological-antitopological fusion, Nucl. Phys. B 367 (1991) 351–461.
[3] V. Cortés, C. Mayer, T. Mohaupt, F. Saueressig, Special geometry of Euclidean supersymmetry I: Vector multiplets, J. High Energy Phys. (JHEP) 0403 (2004) 028, hep-th/0312001.
[4] V. Cortés, L. Schäfer, Topological-antitopological fusion equations, pluriharmonic maps and special Kähler manifolds, in: O. Kowalski, E. Musso, D. Perrone (Eds.), Proceedings of the Conference "Curvature in Geometry", in honor of Lieven Vanhecke, Lecce, in: Progress in Mathematics, vol. 234, Birkhäuser, 2005.
[5] B. Dubrovin, Geometry and integrability of topological-antitopological fusion, Commun. Math. Phys. 152 (1992) 539–564.
[6] D.S. Freed, Special Kähler manifolds, Commun. Math. Phys. 203 (1) (1999) 31–52.
[7] C. Hertling, tt^*-geometry, Frobenius manifolds, their connections, and the construction for singularities, J. Reine Angew. Math. 555 (2003) 77–161.
[8] S. Kobayashi, K. Nomizu, Foundations of Differential Geometry, vols. I, II, Interscience Publishers, 1964, 1969.
[9] L. Schäfer, tt^*-geometry and pluriharmonic maps, Ann. Global Anal. Geom., in press.

Further reading

[10] L. Schäfer, tt^*-Geometrie und pluriharmonische Abbildungen, Diplomarbeit an der Universität Bonn, Dez. 2002.